Lecture notes of the course

Nonsmooth Differential Geometry

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1 Lesson [09/10/2017]

A *metric measure space* is a triple $(\mathrm{X},\mathsf{d},\mathfrak{m})$, where

$$\begin{aligned} (\mathrm{X},\mathsf{d}) \quad & \text{is a complete and separable metric space,} \\ \mathfrak{m}\neq 0 \quad & \text{is a non-negative Borel measure on } (\mathrm{X},\mathsf{d}), \text{ which is finite on balls.} \end{aligned} \tag{1.1}$$

Given a complete and separable metric space (X,d), let us denote

$$\begin{aligned} \mathscr{P}(\mathrm{X}) &:= \text{Borel probability measures on } (\mathrm{X},\mathsf{d}), \\ C_b(\mathrm{X}) &:= \big\{\text{bounded continuous maps } f:\,\mathrm{X}\to\mathbb{R}\big\}. \end{aligned} \tag{1.2}$$

Definition 1.1 (Weak topology) *The* weak topology *on* $\mathscr{P}(\mathrm{X})$ *is defined as the coarsest topology on* $\mathscr{P}(\mathrm{X})$ *such that*

$$\mathscr{P}(\mathrm{X})\ni\mu\longmapsto\int f\,\mathrm{d}\mu \;\; \textit{is continuous,} \quad \textit{for every } f\in C_b(\mathrm{X}). \tag{1.3}$$

Remark 1.2 If a sequence of measures $(\mu_n)_n$ weakly converges to a limit measure μ, then

$$\mu(\Omega)\leq \varliminf_{n\to\infty}\mu_n(\Omega) \quad \text{for every } \Omega\subseteq\mathrm{X} \text{ open.} \tag{1.4}$$

Indeed, let $f_k := k\,\mathsf{d}(\cdot,\mathrm{X}\setminus\Omega)\wedge 1\in C_b(\mathrm{X})$ for $k\in\mathbb{N}$. Hence $f_k(x)\nearrow\chi_\Omega(x)$ for all $x\in\mathrm{X}$, so that $\mu(\Omega)=\sup_k\int f_k\,\mathrm{d}\mu$ by monotone convergence theorem. Since $\nu\mapsto\int f_k\,\mathrm{d}\nu$ is continuous for any k, we deduce that $\nu\mapsto\nu(\Omega)$ is lsc as supremum of continuous maps, yielding (1.4).

In particular, if a sequence $(\mu_n)_n\subseteq\mathscr{P}(\mathrm{X})$ weakly converges to $\mu\in\mathscr{P}(\mathrm{X})$, then

$$\mu(C)\geq \varlimsup_{n\to\infty}\mu_n(C) \quad \text{for every } C\subseteq\mathrm{X} \text{ closed.} \tag{1.5}$$

To prove it, just apply (1.4) to $\Omega:=\mathrm{X}\setminus C$. ■

Remark 1.3 We claim that if $\int f\,\mathrm{d}\mu=\int f\,\mathrm{d}\nu$ for every $f\in C_b(\mathrm{X})$, then $\mu=\nu$.
Indeed, $\mu(C)=\nu(C)$ for any $C\subseteq\mathrm{X}$ closed as a consequence of (1.5), whence $\mu=\nu$ by the monotone class theorem. ■

Remark 1.4 Given any Banach space V, we denote by V' its dual Banach space. Then

$$\mathscr{P}(\mathrm{X}) \;\text{ is continuously embedded into }\; C_b(X)'. \tag{1.6}$$

Such embedding is given by the operator sending $\mu\in\mathscr{P}(\mathrm{X})$ to the map $C_b(\mathrm{X})\ni f\mapsto\int f\,\mathrm{d}\mu$, which is injective by Remark 1.3 and linear by definition. Finally, continuity stems from the inequality $\big|\int f\,\mathrm{d}\mu\big|\leq\|f\|_{C_b(\mathrm{X})}$, which holds for any $f\in C_b(\mathrm{X})$. ■

Fix a countable dense subset $(x_n)_n$ of X. Let us define

$$\begin{aligned} \mathcal{A} &:= \Big\{\big(a-b\,\mathsf{d}(\cdot,x_n)\big)\vee c \;:\; a,b,c\in\mathbb{Q},\, n\in\mathbb{N}\Big\}, \\ \widetilde{\mathcal{A}} &:= \big\{f_1\vee\ldots\vee f_n \;:\; n\in\mathbb{N},\, f_1,\ldots,f_n\in\mathcal{A}\big\}. \end{aligned} \tag{1.7}$$

Observe that $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ are countable subsets of $C_b(\mathrm{X})$. We claim that:

$$f(x) = \sup\big\{g(x)\,:\,g \in \mathcal{A},\, g \leq f\big\} \qquad \text{for every } f \in C_b(\mathrm{X}) \text{ and } x \in \mathrm{X}. \tag{1.8}$$

Indeed, the inequality $\geq$ is trivial, while to prove $\leq$ fix $x \in \mathrm{X}$ and $\varepsilon > 0$. The function f being continuous, there exists a nbhd U of x such that $f(y) \geq f(x) - \varepsilon$ for all $y \in U$. Then we can easily build a function $g \in \mathcal{A}$ such that $g \leq f$ and $g(x) \geq f(x) - 2\,\varepsilon$. By arbitrariness of $x \in \mathrm{X}$ and $\varepsilon > 0$, we thus proved the validity of (1.8).

Exercise 1.5 Suppose that X is compact. If a sequence $(f_n)_n \subseteq C(\mathrm{X})$ satisfies $f_n(x) \searrow 0$ for every $x \in \mathrm{X}$, then $f_n \to 0$ uniformly on X.

Corollary 1.6 *Suppose that* X *is compact. Then* $\widetilde{\mathcal{A}}$ *is dense in* $C(\mathrm{X}) = C_b(\mathrm{X})$. *In particular, the space* $C(\mathrm{X})$ *is separable.*

Proof. Fix $f \in C(\mathrm{X})$. Enumerate $\{g \in \mathcal{A}\,:\,g \leq f\}$ as $(g_n)_n$. Call $h_n := g_1 \vee \ldots \vee g_n \in \widetilde{\mathcal{A}}$ for each $n \in \mathbb{N}$, thus $h_n(x) \nearrow f(x)$ for all $x \in \mathrm{X}$ by (1.8). Hence $(f - h_n)(x) \searrow 0$ for all $x \in \mathrm{X}$ and accordingly $f - h_n \to 0$ in $C(\mathrm{X})$ by Exercise 1.5, proving the thesis. □

Remark 1.7 The converse implication holds: if $C_b(\mathrm{X})$ is separable, then X is compact. ■

Corollary 1.8 *It holds that*

$$\int f\,\mathrm{d}\mu = \sup\left\{\int g\,\mathrm{d}\mu \,\middle|\, g \in \widetilde{\mathcal{A}},\, g \leq f\right\} \qquad \textit{for every } \mu \in \mathscr{P}(\mathrm{X}) \textit{ and } f \in C_b(\mathrm{X}). \tag{1.9}$$

Proof. Call $(g_n)_n = \big\{g \in \mathcal{A}\,:\,g \leq f\big\}$ and put $h_n := g_1 \vee \ldots \vee g_n \in \widetilde{\mathcal{A}}$, thus $h_n(x) \nearrow f(x)$ for all $x \in \mathrm{X}$ and accordingly $\int f\,\mathrm{d}\mu = \lim_n \int h_n\,\mathrm{d}\mu$, proving (1.9). □

We endow $\mathscr{P}(\mathrm{X})$ with a distance δ. Enumerate $\big\{g \in \widetilde{\mathcal{A}} \cap (-\widetilde{\mathcal{A}})\,:\,\|g\|_{C_b(\mathrm{X})} \leq 1\big\}$ as $(f_i)_i$. Then for any $\mu, \nu \in \mathscr{P}(\mathrm{X})$ we define

$$\delta(\mu, \nu) := \sum_{i=0}^{\infty} \frac{1}{2^i} \left|\int f_i\,\mathrm{d}(\mu - \nu)\right|. \tag{1.10}$$

Proposition 1.9 *The weak topology on* $\mathscr{P}(\mathrm{X})$ *is induced by the distance* δ.

Proof. To prove one implication, we want to show that for any $f \in C_b(\mathrm{X})$ the map $\mu \mapsto \int f\,\mathrm{d}\mu$ is δ-continuous. Fix $\mu, \nu \in \mathscr{P}(\mathrm{X})$. Given any $\varepsilon > 0$, there exists a map $g \in \widetilde{\mathcal{A}}$ such that $g \leq f$ and $\int g\,\mathrm{d}\mu \geq \int f\,\mathrm{d}\mu - \varepsilon$, by Corollary 1.8. Let $i \in \mathbb{N}$ be such that $f_i = g/\|g\|_{C_b(\mathrm{X})}$. Then

$$\int f\,\mathrm{d}\nu - \int f\,\mathrm{d}\mu \geq \|g\|_{C_b(\mathrm{X})} \int f_i\,\mathrm{d}(\nu - \mu) - \varepsilon \geq -\|g\|_{C_b(\mathrm{X})}\, 2^i\, \delta(\nu, \mu) - \varepsilon,$$

whence $\varliminf_{\delta(\nu,\mu)\to 0} \int f\,\mathrm{d}(\nu - \mu) \geq 0$ by arbitrariness of $\varepsilon > 0$, i.e. the map $\mu \mapsto \int f\,\mathrm{d}\mu$ is δ-lsc. The δ-upper semicontinuity of $\mu \mapsto \int f\,\mathrm{d}\mu$ can be proved in an analogous way.

Conversely, fix $\mu \in \mathscr{P}(\mathrm{X})$ and $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $2^{-N} < \varepsilon/2$. Then there is a weak nbhd W of μ such that $\big|\int f_i\,\mathrm{d}(\mu - \nu)\big| < \varepsilon/4$ for all $i = 0, \ldots, N$ and $\nu \in W$. Hence

$$\delta(\mu, \nu) \leq \sum_{i=0}^{N} \frac{1}{2^i} \left|\int f_i\,\mathrm{d}(\mu - \nu)\right| + \sum_{i=N+1}^{\infty} \frac{1}{2^i} \leq \frac{\varepsilon}{2} + \frac{1}{2^N} < \varepsilon \qquad \text{for every } \nu \in W,$$

proving that W is contained in the open δ-ball of radius ε centered at μ. □

Remark 1.10 Suppose that X is compact. Then $C(\mathrm{X}) = C_b(\mathrm{X})$, thus accordingly $\mathscr{P}(\mathrm{X})$ is weakly compact by (1.6) and Banach-Alaoglu theorem. Conversely, for X non-compact this is in general no longer true. For instance, take $\mathrm{X} := \mathbb{R}$ and $\mu_n := \delta_n$. Suppose by contradiction that a subsequence $(\mu_{n_m})_m$ weakly converges to some $\mu \in \mathscr{P}(\mathbb{R})$. For any $k \in \mathbb{N}$ we have that $\mu\big((-k,k)\big) \leq \underline{\lim}_m\, \delta_{n_m}\big((-k,k)\big) = 0$, so that $\mu(\mathbb{R}) = \lim_{k\to\infty} \mu\big((-k,k)\big) = 0$, which is absurd. This proves that $\mathscr{P}(\mathbb{R})$ is not weakly compact. ■

Definition 1.11 (Tightness) *A set $\mathcal{K} \subseteq \mathscr{P}(\mathrm{X})$ is said to be* tight *provided for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subseteq \mathrm{X}$ such that $\mu(K_\varepsilon) \geq 1 - \varepsilon$ for every $\mu \in \mathcal{K}$.*

Theorem 1.12 (Prokhorov) *Let $\mathcal{K} \subseteq \mathscr{P}(\mathrm{X})$ be fixed. Then $\mathcal{K}$ is weakly relatively compact if and only if $\mathcal{K}$ is tight.*

Proof. SUFFICIENCY. Fix $\mathcal{K} \subseteq \mathscr{P}(\mathrm{X})$ tight, wlog $\mathcal{K} = (\mu_i)_{i\in\mathbb{N}}$. For any $n \in \mathbb{N}$, choose a compact set $K_n \subseteq \mathrm{X}$ such that $\mu_i(K_n) \geq 1 - 1/n$ for all i. By a diagonalization argument we see that, up to a not relabeled subsequence, $\mu_i|_{K_n}$ converges to some measure ν_n in duality with $C_b(K_n)$ for all $n \in \mathbb{N}$, as a consequence of Remark 1.10. We now claim that:

$$\begin{aligned} &\nu_n \to \nu \ \text{ in total variation norm,} \quad \text{for some measure } \nu,\\ &\mu_i \rightharpoonup \nu \ \text{ in duality with } C_b(\mathrm{X}). \end{aligned} \tag{1.11}$$

To prove the former, recall that for any $m \geq n \geq 1$ one has

$$\|\nu_n - \nu_m\|_{\mathsf{TV}} = \sup\left\{ \int f \,\mathrm{d}(\nu_n - \nu_m) \,\middle|\, f \in C_b(\mathrm{X}),\, \|f\|_{C_b(\mathrm{X})} \leq 1 \right\}.$$

Then fix $f \in C_b(\mathrm{X})$ with $\|f\|_{C_b(\mathrm{X})} \leq 1$. We can assume wlog that $(K_n)_n$ is increasing. We deduce from (1.4) that $\nu_m(K_m \setminus K_n) \leq \underline{\lim}_i\, \mu_i|_{K_m}(\mathrm{X} \setminus K_n) \leq 1/n$. Therefore

$$\int f \,\mathrm{d}(\nu_n - \nu_m) \leq \lim_{i\to\infty} \left(\int_{K_n} f \,\mathrm{d}\mu_i - \int_{K_m} f \,\mathrm{d}\mu_i \right) + \frac{2}{n} \leq \frac{3}{n},$$

proving that $(\nu_n)_n$ is Cauchy wrt $\|\cdot\|_{\mathsf{TV}}$ and accordingly the first in (1.11). For the latter, notice that for any $f \in C_b(\mathrm{X})$ it holds that

$$\begin{aligned} \left| \int f \,\mathrm{d}(\mu_i - \nu) \right| &= \left| \int_{K_n} f \,\mathrm{d}(\mu_i - \nu_n) - \int_{K_n} f \,\mathrm{d}(\nu - \nu_n) + \int_{\mathrm{X}\setminus K_n} f \,\mathrm{d}\mu_i - \int_{\mathrm{X}\setminus K_n} f \,\mathrm{d}\nu \right| \\ &\leq \left| \int_{K_n} f \,\mathrm{d}(\mu_i - \nu_n) \right| + \|f\|_{C_b(\mathrm{X})} \|\nu - \nu_n\|_{\mathsf{TV}} + \frac{2\,\|f\|_{C_b(\mathrm{X})}}{n}. \end{aligned}$$

By first letting $i \to \infty$ and then $n \to \infty$, we obtain that $\lim_i \left| \int f \,\mathrm{d}(\mu_i - \nu) \right| = 0$, showing the second in (1.11). Hence sufficiency is proved.
NECESSITY. Fix $\mathcal{K} \subseteq \mathscr{P}(\mathrm{X})$ weakly relatively compact. Choose $\varepsilon > 0$ and a sequence $(x_n)_n$ that is dense in X. Arguing by contradiction, we aim to prove that

$$\forall i \in \mathbb{N} \quad \exists N_i \in \mathbb{N}: \quad \mu\bigg(\bigcup_{j=1}^{N_i} \bar{B}_{1/i}(x_j) \bigg) \geq 1 - \frac{\varepsilon}{2^i} \quad \forall \mu \in \mathcal{K}. \tag{1.12}$$

If not, there exist $i_0 \in \mathbb{N}$ and $(\mu_m)_m \subseteq \mathcal{K}$ such that $\mu_m\big(\bigcup_{j=1}^m \bar{B}_{1/i_0}(x_j)\big) < 1-\varepsilon$ holds for every $m \in \mathbb{N}$. Up to a not relabeled subsequence $\mu_m \rightharpoonup \mu \in \mathscr{P}(\mathrm{X})$ and accordingly

$$\mu\bigg(\bigcup_{j=1}^n B_{1/i_0}(x_j)\bigg) \overset{(1.4)}{\leq} \varliminf_{m\to\infty} \mu_m\bigg(\bigcup_{j=1}^m \bar{B}_{1/i_0}(x_j)\bigg) \leq 1-\varepsilon \quad \text{for any } n \in \mathbb{N},$$

which contradicts the fact that $\lim_{n\to\infty} \mu\big(\bigcup_{j=1}^n B_{1/i_0}(x_j)\big) = \mu(\mathrm{X}) = 1$. This proves (1.12).

Now define $K := \bigcap_{i\in\mathbb{N}} \bigcup_{j=1}^{N_i} \bar{B}_{1/i}(x_j)$. Such set is compact, as it is closed and totally bounded by construction. Moreover, for any $\mu \in \mathcal{K}$ one has that

$$\mu(\mathrm{X}\setminus K) \leq \sum_i \mu\bigg(\bigcap_{j=1}^{N_i} \mathrm{X}\setminus \bar{B}_{1/i}(x_j)\bigg) \overset{(1.12)}{\leq} \varepsilon \sum_i \frac{1}{2^i} = \varepsilon,$$

proving also necessity. □

Remark 1.13 We have that a set $\mathcal{K} \subseteq \mathscr{P}(\mathrm{X})$ is tight if and only if

$$\exists\, \Psi : \mathrm{X} \to [0,+\infty], \text{ with compact sublevels, such that } s := \sup_{\mu\in\mathcal{K}} \int \Psi \,\mathrm{d}\mu < +\infty. \tag{1.13}$$

To prove sufficiency, first notice that Ψ is Borel as its sublevels are closed sets. Now fix $\varepsilon > 0$ and choose $C > 0$ such that $s/C < \varepsilon$. Moreover, by applying Čebyšëv's inequality we obtain that $C\,\mu\{\Psi > C\} \leq \int \Psi \,\mathrm{d}\mu \leq s$ for all $\mu \in \mathcal{K}$, whence $\mu\big(\{\Psi \leq C\}\big) \geq 1 - s/C > 1-\varepsilon$.

To prove necessity, suppose $\mathcal{K}$ tight and choose a sequence $(K_n)_n$ of compact sets such that $\mu(\mathrm{X}\setminus K_n) \leq 1/n^3$ for all $n \in \mathbb{N}$ and $\mu \in \mathcal{K}$. Define $\Psi(x) := \inf\big\{n \in \mathbb{N} : x \in K_n\big\}$ for every $x \in \mathrm{X}$. Clearly Ψ has compact sublevels by construction. Moreover, it holds that

$$\sup_{\mu\in\mathcal{K}} \int \Psi \,\mathrm{d}\mu = \sup_{\mu\in\mathcal{K}} \sum_n \int_{K_{n+1}\setminus K_n} \Psi \,\mathrm{d}\mu \leq \sum_n \frac{n+1}{n^3} < +\infty,$$

as required. ■

2 Lesson [11/10/2017]

Remark 2.1 Let $\mu \geq 0$ be a finite Borel measure on X. Then for any $E \subseteq \mathrm{X}$ Borel one has

$$\mu(E) = \sup\big\{\mu(C) : C \subseteq E \text{ closed}\big\} = \inf\big\{\mu(\Omega) : \Omega \supseteq E \text{ open}\big\}. \tag{2.1}$$

To prove it, it suffices to show that the family of all Borel sets E satisfying (2.1), which we shall denote by $\mathcal{E}$, forms a σ-algebra containing all open subsets of X. Then fix $\Omega \subseteq \mathrm{X}$ open. Call $C_n := \big\{x \in \Omega : \mathsf{d}(x, \mathrm{X}\setminus\Omega) \geq 1/n\big\}$ for all $n \in \mathbb{N}$, whence $(C_n)_n$ is an increasing sequence of closed sets and $\mu(\Omega) = \lim_n \mu(C_n)$ by continuity from below of μ. This grants that $\Omega \in \mathcal{E}$.

It only remains to show that $\mathcal{E}$ is a σ-algebra. It is obvious that $\emptyset \in \mathcal{E}$ and that $\mathcal{E}$ is stable by complements. Now fix $(E_n)_n \subseteq \mathcal{E}$ and $\varepsilon > 0$. There exist $(C_n)_n$ closed and $(\Omega_n)_n$ open such that $C_n \subseteq E_n \subseteq \Omega_n$ and $\mu(\Omega_n) - \varepsilon\, 2^{-n} \leq \mu(E_n) \leq \mu(C_n) + \varepsilon\, 2^{-n}$ for every $n \in \mathbb{N}$. Let us denote $\Omega := \bigcup_n \Omega_n$. Moreover, continuity from above of μ yields the existence of $N \in \mathbb{N}$ such that $\mu\big(\bigcup_{n\in\mathbb{N}} C_n \setminus C\big) \le \varepsilon$, where we put $C := \bigcup_{n=1}^N C_n$. Notice that Ω is open, C is closed and $C \subseteq \bigcup_n E_n \subseteq \Omega$. Finally, it holds that

$$\mu\bigg(\bigcup_{n=1}^{\infty} E_n \setminus C\bigg) \le \sum_{n=1}^{\infty} \mu(E_n \setminus C_n) + \varepsilon \le \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} + \varepsilon = 2\,\varepsilon,$$
$$\mu\bigg(\Omega \setminus \bigcup_{n=1}^{\infty} E_n\bigg) \le \sum_{n=1}^{\infty} \mu(\Omega_n \setminus E_n) \le \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

This grants that $\bigcup_n E_n \in \mathcal{E}$, concluding the proof. ■

Remark 2.2 (Total variation norm) During the proof of Theorem 1.12, we needed the following two properties of the *total variation norm*:

$$\|\mu\|_{\mathsf{TV}} = \sup\bigg\{\int f\,\mathrm{d}\mu \,\bigg|\, f \in C_b(\mathrm{X}),\ \|f\|_{C_b(\mathrm{X})} \le 1\bigg\} \quad \text{for any signed Borel measure } \mu \text{ on } \mathrm{X}, \tag{2.2}$$
$$\big(\mathscr{P}(\mathrm{X}), \|\cdot\|_{\mathsf{TV}}\big) \quad \text{is complete.}$$

In order to prove them, we proceed as follows. Given a signed measure μ, let us consider its *Hahn-Jordan decomposition* $\mu = \mu^+ - \mu^-$, where $\mu^\pm$ are non-negative measures with $\mu^+ \perp \mu^-$, which satisfy $\mu(P) = \mu^+(\mathrm{X})$ and $\mu(P^c) = -\mu^-(\mathrm{X})$ for a suitable Borel set $P \subseteq \mathrm{X}$. Hence by definition the total variation norm is defined as

$$\|\mu\|_{\mathsf{TV}} := \mu^+(\mathrm{X}) + \mu^-(\mathrm{X}). \tag{2.3}$$

Such definition is well-posed, since the Hahn-Jordan decomposition (μ^+, μ^-) of μ is unique.

To prove the first in (2.2), we start by noticing that $\int f\,\mathrm{d}\mu \le \int |f|\,\mathrm{d}(\mu^+ + \mu^-) \le \|\mu\|_{\mathsf{TV}}$ holds for any $f \in C_b(\mathrm{X})$ with $\|f\|_{C_b(\mathrm{X})} \le 1$, proving one inequality. To show the converse one, let $\varepsilon > 0$ be fixed. By Remark 2.1, we can choose two closed sets $C \subseteq P$ and $C' \subseteq P^c$ such that $\mu^+(P \setminus C), \mu^-(P^c \setminus C') < \varepsilon$. Call $f_n := \big(1 - n\,\mathsf{d}(\cdot, C)\big)^+$ and $g_n := \big(1 - n\,\mathsf{d}(\cdot, C')\big)^+$, so that $f_n \searrow \chi_C$ and $g_n \searrow \chi_{C'}$ as $n \to \infty$. Now define $h_n := f_n - g_n$. Since $|h_n| \le 1$, we have that $(h_n)_n \subseteq C_b(\mathrm{X})$ and $\|h_n\|_{C_b(\mathrm{X})} \le 1$ for every $n \in \mathbb{N}$. Moreover, it holds that

$$\begin{aligned}\lim_{n\to\infty} \int h_n\,\mathrm{d}\mu &= \lim_{n\to\infty} \bigg[\int f_n\,\mathrm{d}\mu^+ - \int f_n\,\mathrm{d}\mu^- - \int g_n\,\mathrm{d}\mu^+ + \int g_n\,\mathrm{d}\mu^-\bigg] \\ &= \mu^+(C) + \mu^-(C') \ge \mu^+(P) + \mu^-(P^c) - 2\,\varepsilon = \|\mu\|_{\mathsf{TV}} - 2\,\varepsilon.\end{aligned}$$

By arbitrariness of $\varepsilon > 0$, we conclude that $\lim_n \int h_n\,\mathrm{d}\mu = \|\mu\|_{\mathsf{TV}}$, proving the first in (2.2).

To show the second, fix a sequence $(\mu_n)_n \subseteq \mathscr{P}(\mathrm{X})$ that is $\|\cdot\|_{\mathsf{TV}}$-Cauchy. Notice that

$$\big|\mu(E)\big| \le \|\mu\|_{\mathsf{TV}} \quad \text{for every signed measure } \mu \text{ and Borel set } E \subseteq \mathrm{X}.$$

Indeed, $\big|\mu(E)\big| \le \mu^+(E) + \mu^-(E) \le \mu^+(\mathrm{X}) + \mu^-(\mathrm{X}) = \|\mu\|_{\mathsf{TV}}$. Therefore

$$\big|\mu_n(E) - \mu_m(E)\big| \le \|\mu_n - \mu_m\|_{\mathsf{TV}} \quad \text{for every } n, m \in \mathbb{N} \text{ and } E \subseteq \mathrm{X} \text{ Borel.} \tag{2.4}$$

In particular, $\big(\mu_n(E)\big)_n$ is Cauchy for any $E \subseteq \mathrm{X}$ Borel, so that $\lim_n \mu_n(E) = \mathsf{L}(E)$ for some limit $\mathsf{L}(E) \in [0,1]$. We thus deduce from (2.4) that

$$\forall \varepsilon > 0 \quad \exists \bar{n}_\varepsilon \in \mathbb{N} : \quad \big|\mathsf{L}(E) - \mu_n(E)\big| \leq \varepsilon \quad \forall n \geq \bar{n}_\varepsilon \quad \forall E \subseteq \mathrm{X} \text{ Borel.} \tag{2.5}$$

We claim that L is a measure. Clearly, $\mathsf{L}(\emptyset) = 0$ and $\mathsf{L}(\mathrm{X}) = 1$. For E, F Borel with $E \cap F = \emptyset$, we have that $\mathsf{L}(E \cup F) = \lim_n \mu_n(E \cup F) = \lim_n \mu_n(E) + \lim_n \mu_n(F) = \mathsf{L}(E) + \mathsf{L}(F)$, which grants that L is finitely additive. To show that it is also σ-additive, fix any sequence $(E_i)_i$ of pairwise disjoint Borel sets. Let us call $U_N := \bigcup_{i=1}^N E_i$ for all $N \in \mathbb{N}$ and $U := \bigcup_{i=1}^\infty E_i$. Given any $\varepsilon > 0$, we infer from (2.5) that for any $n \geq \bar{n}_\varepsilon$ one has

$$\begin{aligned}
\varlimsup_{N \to \infty} \big|\mathsf{L}(U) - \mathsf{L}(U_N)\big| &\leq \big|\mathsf{L}(U) - \mu_n(U)\big| + \varlimsup_{N \to \infty} \big|\mu_n(U) - \mu_n(U_N)\big| + \varlimsup_{N \to \infty} \big|\mu_n(U_N) - \mathsf{L}(U_N)\big| \\
&\leq 2\,\varepsilon + \varlimsup_{N \to \infty} \big|\mu_n(U) - \mu_n(U_N)\big| = 2\,\varepsilon,
\end{aligned}$$

where the last equality follows from the continuity from above of μ_n. By letting $\varepsilon \to 0$ in the previous formula, we thus obtain that $\mathsf{L}(U) = \lim_N \mathsf{L}(U_N) = \sum_{i=1}^\infty \mathsf{L}(E_i)$, so that $\mathsf{L} \in \mathscr{P}(\mathrm{X})$. Finally, we aim to prove that $\lim_n \|\mathsf{L} - \mu_n\|_{\mathsf{TV}} = 0$. For any $n \in \mathbb{N}$, choose a Borel set $P_n \subseteq \mathrm{X}$ satisfying $(\mathsf{L} - \mu_n)(P_n) = (\mathsf{L} - \mu_n)^+(\mathrm{X})$ and $(\mathsf{L} - \mu_n)(P_n^c) = -(\mathsf{L} - \mu_n)^-(\mathrm{X})$. Now fix $\varepsilon > 0$. Hence (2.5) guarantees that for every $n \geq \bar{n}_\varepsilon$ it holds that

$$\|\mathsf{L} - \mu_n\|_{\mathsf{TV}} = (\mathsf{L} - \mu_n)(P_n) - (\mathsf{L} - \mu_n)(P_n^c) = \big|(\mathsf{L} - \mu_n)(P_n)\big| + \big|(\mathsf{L} - \mu_n)(P_n^c)\big| \leq 2\,\varepsilon.$$

Therefore μ_n converges to L in the $\|\cdot\|_{\mathsf{TV}}$-norm, concluding the proof of (2.2). ■

We now present some consequences of Theorem 1.12:

Corollary 2.3 (Ulam's theorem) *Any $\mu \in \mathscr{P}(\mathrm{X})$ is concentrated on a σ-compact set.*

Proof. Clearly the singleton $\{\mu\}$ is weakly relatively compact, so it is tight by Theorem 1.12. Thus for any $n \in \mathbb{N}$ we can choose a compact set $K_n \subseteq \mathrm{X}$ such that $\mu(\mathrm{X} \setminus K_n) < 1/n$. In particular, μ is concentrated on $\bigcup_n K_n$, yielding the thesis. □

Corollary 2.4 *Let $\mu \in \mathscr{P}(\mathrm{X})$ be given. Then μ is* inner regular, *i.e.*

$$\mu(E) = \sup\big\{\mu(K) \,:\, K \subseteq E \textit{ compact}\big\} \quad \textit{for every } E \subseteq \mathrm{X} \textit{ Borel.} \tag{2.6}$$

In particular, μ is a Radon measure.

Proof. By Corollary 2.3, there exists an increasing sequence $(K_n)_n$ of compact subsets of X such that $\lim_n \mu\big(\mathrm{X} \setminus K_n\big) = 0$. Any closed subset C of X that is contained in some K_n is clearly compact, whence

$$\begin{aligned}
\mu(E) = \lim_{n \to \infty} \mu(E \cap K_n) &= \lim_{n \to \infty} \sup\big\{\mu(C) \,:\, C \subseteq E \cap K_n \text{ closed}\big\} \\
&\leq \sup\big\{\mu(K) \,:\, K \subseteq E \text{ compact}\big\} \qquad \text{for every } E \subseteq \mathrm{X} \text{ Borel,}
\end{aligned}$$

proving (2.6), as required. □

Given any function $f : \mathrm{X} \to \mathbb{R}$, let us define

$$\mathrm{Lip}(f) := \sup_{\substack{x,y\in\mathrm{X}\\ x\neq y}} \frac{\big|f(x)-f(y)\big|}{\mathsf{d}(x,y)} \in [0,+\infty]. \tag{2.7}$$

We say that f is *Lipschitz* provided $\mathrm{Lip}(f) < +\infty$ and we define

$$\begin{aligned} \mathrm{LIP}(\mathrm{X}) &:= \big\{ f : \mathrm{X} \to \mathbb{R} \;:\; \mathrm{Lip}(f) < +\infty \big\}, \\ \mathrm{LIP}_{bs}(\mathrm{X}) &:= \big\{ f \in \mathrm{LIP}(\mathrm{X}) \;:\; \mathrm{spt}(f) \text{ is bounded} \big\} \subseteq C_b(\mathrm{X}). \end{aligned} \tag{2.8}$$

We point out that continuous maps having bounded support are not necessarily bounded.

Proposition 2.5 (Separability of $L^p(\mu)$ for $p<\infty$) *Let $\mu \in \mathscr{P}(\mathrm{X})$ and $p \in [1,\infty)$. Then the space $\mathrm{LIP}_{bs}(\mathrm{X})$ is dense in $L^p(\mu)$. In particular, the space $L^p(\mu)$ is separable.*

Proof. First, notice that $\mathrm{LIP}_{bs}(\mathrm{X}) \subseteq L^\infty(\mu) \subseteq L^p(\mu)$. Call $\mathscr{C}$ the $L^p(\mu)$-closure of $\mathrm{LIP}_{bs}(\mathrm{X})$.
Step 1. We claim that $\big\{\chi_C \;:\; C \subseteq \mathrm{X} \text{ closed bounded}\big\}$ is contained in the set $\mathscr{C}$. Indeed, called $f_n := \big(1 - n\,\mathsf{d}(\cdot,C)\big)^+ \in \mathrm{LIP}_{bs}(\mathrm{X})$ for any $n \in \mathbb{N}$, one has $f_n \to \chi_C$ in $L^p(\mu)$ by dominated convergence theorem.
Step 2. We also have that $\big\{\chi_E \;:\; E \subseteq \mathrm{X} \text{ Borel}\big\} \subseteq \mathscr{C}$. Indeed, we can pick an increasing sequence $(C_n)_n$ of closed subsets of E such that $\mu(E) = \lim_n \mu(C_n)$, as seen in (2.1). Then one has that $\|\chi_E - \chi_{C_n}\|_{L^p(\mu)} = \mu(E \setminus C_n)^{1/p} \to 0$, whence $\chi_E \in \mathscr{C}$ by Step 1.
Step 3. To prove that $L^p(\mu) \subseteq \mathscr{C}$, fix $f \in L^p(\mu)$, wlog $f \geq 0$. Given any $n, i \in \mathbb{N}$, let us define $E_{ni} := f^{-1}\big([i/2^n,(i+1)/2^n[\big)$. Observe that $(E_{ni})_i$ is a Borel partition of X, thus it makes sense to define $f_n := \sum_{i\in\mathbb{N}} i\,2^{-n}\,\chi_{E_{ni}} \in L^p(\mu)$. Since $f_n(x) \nearrow f(x)$ for μ-a.e. $x \in \mathrm{X}$, we have $f_n \to f$ in $L^p(\mu)$ by dominated convergence theorem. We aim to prove that $(f_n)_n \subseteq \mathscr{C}$, which would immediately imply that $f \in \mathscr{C}$. Then fix $n \in \mathbb{N}$. Notice that f_n is the $L^p(\mu)$-limit of $f_n^N := \sum_{i=1}^N i\,2^{-n}\,\chi_{E_{ni}}$ as $N \to \infty$, again by dominated convergence theorem. Given that each $f_n^N \in \mathscr{C}$ by Step 2, we get that f_n is in $\mathscr{C}$ as well. Hence $\mathrm{LIP}_{bs}(\mathrm{X})$ is dense in $L^p(\mu)$.
Step 4. Finally, we prove separability of $L^p(\mu)$. We can take an increasing sequence $(K_n)_n$ of compact subsets of X such that the measure μ is concentrated on $\bigcup_n K_n$, by Corollary 2.3. Since $\chi_{K_n} f \to f$ in $L^p(\mu)$ for any $f \in L^p(\mu)$, we see that

$$\bigcup_{n\in\mathbb{N}} \underbrace{\big\{ f \in L^p(\mu) \;:\; f = 0 \;\; \mu\text{-a.e. in } \mathrm{X}\setminus K_n \big\}}_{=:S_n} \quad \text{is dense in } L^p(\mu).$$

To conclude, it is sufficient to show that each S_n is separable. Observe that $C(K_n)$ is separable by Corollary 1.6, thus accordingly its subset $\mathrm{LIP}_{bs}(K_n)$ is separable with respect to $\|\cdot\|_{C_b(K_n)}$. In particular, $\mathrm{LIP}_{bs}(K_n)$ is separable with respect to $\|\cdot\|_{L^p(\mu)}$. Moreover, $\mathrm{LIP}_{bs}(K_n)$ is dense in $L^p(\mu_{|K_n}) \cong S_n$ by the first part of the statement, therefore each S_n is separable. ☐

3 Lesson [16/10/2017]

We equip the space $C([0,1],\mathrm{X})$ of all continuous curves in X with the *sup distance*:

$$\underline{\mathsf{d}}(\gamma,\tilde\gamma) := \max_{t\in[0,1]} \mathsf{d}(\gamma_t,\tilde\gamma_t) \qquad \text{for every } \gamma,\tilde\gamma \in C([0,1],\mathrm{X}). \tag{3.1}$$

Proposition 3.1 *Let* (X, d) *be a complete (resp. separable) metric space. Then the metric space* $\big(C([0,1],\mathrm{X}), \underline{\mathsf{d}}\big)$ *is complete (resp. separable).*

Proof. COMPLETENESS. Take a $\underline{\mathsf{d}}$-Cauchy sequence $(\gamma^n)_n \subseteq C([0,1],\mathrm{X})$. Hence for any $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $\underline{\mathsf{d}}(\gamma^n, \gamma^m) < \varepsilon$ for all $n, m \geq n_\varepsilon$. In particular, $(\gamma^n_t)_n$ is d-Cauchy for each $t \in [0,1]$, so that $\lim_n \gamma^n_t = \gamma_t$ with respect to d for a suitable $\gamma_t \in \mathrm{X}$, by completeness of (X, d). Given any $\varepsilon > 0$ and $n \geq n_\varepsilon$, we have $\sup_t \mathsf{d}(\gamma^n_t, \gamma_t) \leq \sup_t \lim_m \mathsf{d}(\gamma^n_t, \gamma^m_t) \leq \varepsilon$ and

$$\varlimsup_{s\to t} \mathsf{d}(\gamma_s, \gamma_t) \leq \varlimsup_{s\to t} \big[\mathsf{d}(\gamma_s, \gamma^n_s) + \mathsf{d}(\gamma^n_s, \gamma^n_t) + \mathsf{d}(\gamma^n_t, \gamma_t)\big] \leq 2\,\varepsilon + \lim_{s\to t} \mathsf{d}(\gamma^n_s, \gamma^n_t) = 2\,\varepsilon \quad \forall t \in [0,1],$$

proving that γ is continuous and $\lim_n \underline{\mathsf{d}}(\gamma^n, \gamma) = 0$. Then $\big(C([0,1],\mathrm{X}), \underline{\mathsf{d}}\big)$ is complete.
SEPARABILITY. Fix $(x_n)_n \subseteq \mathrm{X}$ dense. Given $k, n \in \mathbb{N}$ and $f : \{0, \ldots, n-1\} \to \mathbb{N}$, we let

$$A_{k,n,f} := \Big\{\gamma \in C([0,1],\mathrm{X}) \,\Big|\, \mathsf{d}(\gamma_t, x_{f(i)}) < 1/2^k \quad \forall i = 0, \ldots, n-1,\ t \in \big[i/n, (i+1)/n\big]\Big\}.$$

We then claim that

$$\begin{aligned} &\bigcup_{n,f} A_{k,n,f} = C([0,1],\mathrm{X}) \quad \text{ for every } k \in \mathbb{N}, \\ &\underline{\mathsf{d}}(\gamma, \tilde\gamma) \leq \frac{1}{2^{k-1}} \quad \text{ for every } \gamma, \tilde\gamma \in A_{k,n,f}. \end{aligned} \tag{3.2}$$

To prove the first in (3.2), fix $k \in \mathbb{N}$ and $\gamma \in C([0,1],\mathrm{X})$. Since γ is uniformly continuous, there exists $\delta > 0$ such that $\mathsf{d}(\gamma_t, \gamma_s) < 1/2^{k+1}$ provided $t, s \in [0,1]$ satisfy $|t - s| < \delta$. Choose any $n \in \mathbb{N}$ such that $1/n < \delta$. Since $(x_n)_n$ is dense in X, for every $i = 0, \ldots, n-1$ we can choose $f(i) \in \mathbb{N}$ such that $\mathsf{d}(x_{f(i)}, \gamma_{i/n}) < 1/2^{k+1}$. Hence for any $i = 0, \ldots, n-1$ it holds that

$$\mathsf{d}(\gamma_t, x_{f(i)}) \leq \mathsf{d}(\gamma_t, \gamma_{i/n}) + \mathsf{d}(\gamma_{i/n}, x_{f(i)}) < \frac{1}{2^k} \quad \text{ for every } t \in \left[\frac{i}{n}, \frac{i+1}{n}\right],$$

proving that $\gamma \in A_{k,n,f}$ and accordingly the first in (3.2). To prove the second, simply notice that $\mathsf{d}(\gamma_t, \tilde\gamma_t) \leq \mathsf{d}(\gamma_t, x_{f(i)}) + \mathsf{d}(x_{f(i)}, \tilde\gamma_t) < 1/2^{k-1}$ for all $i = 1, \ldots, n-1$ and $t \in \big[i/n, (i+1)/n\big]$.

In order to conclude, pick any $\gamma^{k,n,f} \in A_{k,n,f}$ for every k, n, f. The family $(\gamma^{k,n,f})_{k,n,f}$, which is countable by construction, is $\underline{\mathsf{d}}$-dense in $C([0,1],\mathrm{X})$ by (3.2), giving the thesis. □

We say that $C([0,1],\mathrm{X})$ is a *Polish space*, i.e. a topological space whose topology comes from a complete and separable distance.

Remark 3.2 Any open subset of a Polish space is a Polish space. ■

Exercise 3.3 Given any two topological spaces Y and Z, we define the *compact-open topology* on $C(\mathrm{Y}, \mathrm{Z})$ as follows: for $K \subseteq \mathrm{Y}$ compact and $\Omega \subseteq \mathrm{Z}$ open we denote

$$\mathcal{V}_{K,\Omega} := \big\{f \in C(\mathrm{Y}, \mathrm{Z}) \;:\; f(K) \subseteq \Omega\big\},$$

then the compact-open topology is defined as the one that is generated by all $\mathcal{V}_{K,\Omega}$.

Prove that $\underline{\mathsf{d}}$ induces the compact-open topology on $C([0,1],\mathrm{X})$.

Definition 3.4 (Absolutely continuous curves) *We say that a curve* $\gamma : [0,1] \to \mathrm{X}$ *is* absolutely continuous*, briefly* AC*, provided there exists a map* $f \in L^1(0,1)$ *such that*

$$\mathsf{d}(\gamma_t, \gamma_s) \leq \int_s^t f(r)\,\mathrm{d}r \quad \text{for every } t, s \in [0,1] \text{ with } s < t. \tag{3.3}$$

Clearly, all absolutely continuous curves are continuous.

Remark 3.5 If $\mathrm{X} = \mathbb{R}$ then this notion of AC curve coincides with the classical one. ■

Theorem 3.6 (Metric speed) *Let* γ *be an absolutely continuous curve in* X*. Then*

$$\exists\,|\dot\gamma_t| := \lim_{h \to 0} \frac{\mathsf{d}(\gamma_{t+h}, \gamma_t)}{|h|} \quad \text{for a.e. } t \in [0,1]. \tag{3.4}$$

Moreover, the function $|\dot\gamma|$*, which is called* metric speed *of* γ*, belongs to* $L^1(0,1)$ *and is the minimal function (in the a.e. sense) that can be chosen as* f *in* (3.3)*.*

Proof. Fix $(x_n)_n \subseteq \mathrm{X}$ dense. We define $g_n(t) := \mathsf{d}(\gamma_t, x_n)$ for all $t \in [0,1]$. Then

$$\big|g_n(t) - g_n(s)\big| \leq \mathsf{d}(\gamma_t, \gamma_s) \leq \int_s^t f(r)\,\mathrm{d}r \quad \text{for every } t, s \in [0,1] \text{ with } s < t, \tag{3.5}$$

showing that each $g_n : [0,1] \to \mathbb{R}$ is AC. Hence g_n is differentiable a.e. and by applying the Lebesgue differentiation theorem to (3.5) we get that $\big|g_n'(t)\big| \leq f(t)$ for a.e. $t \in [0,1]$. Let us call $g := \sup_n g_n'$, so that $g \in L^1(0,1)$ with $|g| \leq f$ a.e.. Moreover, one has that

$$\mathsf{d}(\gamma_t, \gamma_s) = \sup_{n \in \mathbb{N}} \big[g_n(t) - g_n(s)\big] \quad \text{for every } t, s \in [0,1]. \tag{3.6}$$

Indeed, $\mathsf{d}(\gamma_t, \gamma_s) \geq \big[g_n(t) - g_n(s)\big]$ for all n by triangle inequality. On the other hand, given any $\varepsilon > 0$ we can choose $n \in \mathbb{N}$ such that $\mathsf{d}(x_n, \gamma_s) < \varepsilon$, whence $g_n(t) - g_n(s) \geq \mathsf{d}(\gamma_t, \gamma_s) - 2\,\varepsilon$.

We thus deduce from (3.6) that g can substitute the function f in (3.3), because

$$\mathsf{d}(\gamma_t, \gamma_s) = \sup_{n \in \mathbb{N}} \int_s^t g_n'(r)\,\mathrm{d}r \leq \int_s^t g(r)\,\mathrm{d}r \quad \text{for every } t, s \in [0,1] \text{ with } s < t. \tag{3.7}$$

In order to conclude, it only remains to prove that g is actually the metric speed. By applying Lebesgue differentiation theorem to (3.7), we see that $\overline{\lim}_{s \to t} \mathsf{d}(\gamma_t, \gamma_s)/|t-s| \leq g(t)$ holds for almost every $t \in [0,1]$. Conversely, $\mathsf{d}(\gamma_t, \gamma_s) \geq g_n(t) - g_n(s) = \int_s^t g_n'(r)\,\mathrm{d}r$ holds for all $s < t$ and $n \in \mathbb{N}$ by triangle inequality, so $\underline{\lim}_{s \to t} \mathsf{d}(\gamma_t, \gamma_s)/|t-s| \geq g_n'(t)$ is satisfied for a.e. $t \in [0,1]$ and for every $n \in \mathbb{N}$ by Lebesgue differentiation theorem. This implies that

$$g(t) \geq \overline{\lim_{s \to t}} \frac{\mathsf{d}(\gamma_t, \gamma_s)}{|t-s|} \geq \underline{\lim_{s \to t}} \frac{\mathsf{d}(\gamma_t, \gamma_s)}{|t-s|} \geq \sup_{n \in \mathbb{N}} g_n'(t) = g(t) \quad \text{for a.e. } t \in [0,1],$$

thus concluding the proof. □

We define the *kinetic energy functional* $\mathrm{KE}: C([0,1],\mathrm{X}) \to [0,+\infty]$ as follows:

$$\mathrm{KE}(\gamma) := \begin{cases} \int_0^1 |\dot\gamma_t|^2 \,\mathrm{d}t & \text{if } \gamma \text{ is AC,} \\ +\infty & \text{if } \gamma \text{ is not AC.} \end{cases} \tag{3.8}$$

Proposition 3.7 *The functional* KE *is* $\underline{\mathsf{d}}$*-lower semicontinuous.*

Proof. Fix a sequence $(\gamma^n)_n \subseteq C([0,1],\mathrm{X})$ that $\underline{\mathsf{d}}$-converges to some $\gamma \in C([0,1],\mathrm{X})$. We can take a subsequence $(\gamma^{n_k})_k$ satisfying $\lim_k \mathrm{KE}(\gamma^{n_k}) = \underline{\lim}_n \mathrm{KE}(\gamma^n)$. Our aim is to prove the inequality $\mathrm{KE}(\gamma) \leq \lim_k \mathrm{KE}(\gamma^{n_k})$. The case in which $\lim_k \mathrm{KE}(\gamma^{n_k}) = +\infty$ is trivial, so suppose that such limit is finite. In particular, up to discarding finitely many γ^{n_k}'s, we have that all curves γ^{n_k} are absolutely continuous with $\big(|\dot\gamma^{n_k}|\big)_k \subseteq L^2(0,1)$ bounded. Therefore, up to a not relabeled subsequence, $|\dot\gamma^{n_k}|$ converges to some limit function $G \in L^2(0,1) \subseteq L^1(0,1)$ weakly in $L^2(0,1)$. Given any $t,s \in [0,1]$ with $s<t$, we thus have that

$$\mathsf{d}(\gamma_t,\gamma_s) = \lim_{k\to\infty} \mathsf{d}(\gamma^{n_k}_t,\gamma^{n_k}_s) \leq \underline{\lim}_{k\to\infty} \int_s^t |\dot\gamma^{n_k}_r| \,\mathrm{d}r = \lim_{k\to\infty} \big\langle |\dot\gamma^{n_k}|, \chi_{[s,t]} \big\rangle_{L^2(0,1)} = \int_s^t G(r)\,\mathrm{d}r,$$

which grants that γ is absolutely continuous with $|\dot\gamma| \leq G$ a.e. by Theorem 3.6. Hence

$$\mathrm{KE}(\gamma) = \int_0^1 |\dot\gamma_t|^2 \,\mathrm{d}t \leq \|G\|^2_{L^2(0,1)} \leq \underline{\lim}_{k\to\infty} \int_0^1 |\dot\gamma^{n_k}_t|^2 \,\mathrm{d}t = \lim_{k\to\infty} \mathrm{KE}(\gamma^{n_k}),$$

proving the thesis. □

Exercise 3.8 Prove that

$$\mathrm{KE}(\gamma) = \sup_{0=t_0<\ldots<t_n=1} \sum_{i=0}^{n-1} \frac{\mathsf{d}(\gamma_{t_{i+1}},\gamma_{t_i})^2}{t_{i+1}-t_i} \quad \text{holds for every } \gamma \in C([0,1],\mathrm{X}). \tag{3.9}$$

Definition 3.9 (Geodesic curve) *A curve* $\gamma : [0,1] \to \mathrm{X}$ *is said to be a* geodesic *provided*

$$\mathsf{d}(\gamma_t,\gamma_s) \leq |t-s|\,\mathsf{d}(\gamma_0,\gamma_1) \quad \textit{holds for every } t,s \in [0,1]. \tag{3.10}$$

Clearly, any geodesic curve is continuous.

Proposition 3.10 *Let* $\gamma \in C([0,1],\mathrm{X})$ *be fixed. Then the following are equivalent:*

i) γ *is a geodesic,*

ii) $\mathsf{d}(\gamma_t,\gamma_s) = |t-s|\,\mathsf{d}(\gamma_0,\gamma_1)$ *for every* $t,s \in [0,1]$,

iii) γ *is AC, its metric speed* $|\dot\gamma|$ *is a.e. constant and* $\mathsf{d}(\gamma_0,\gamma_1) = \int_0^1 |\dot\gamma_t|\,\mathrm{d}t$,

iv) $\mathrm{KE}(\gamma) = \mathsf{d}(\gamma_0,\gamma_1)^2$.

Proof. i) $\implies$ ii) Suppose that $\mathsf{d}(\gamma_t, \gamma_s) < (t-s)\,\mathsf{d}(\gamma_0, \gamma_1)$ for some $0 \le s < t \le 1$, then

$$\mathsf{d}(\gamma_0, \gamma_1) \le \mathsf{d}(\gamma_0, \gamma_s) + \mathsf{d}(\gamma_s, \gamma_t) + \mathsf{d}(\gamma_t, \gamma_1) < \big[t + (t-s) + (1-s)\big]\,\mathsf{d}(\gamma_0, \gamma_1) = \mathsf{d}(\gamma_0, \gamma_1),$$

which leads to contradiction. Hence $\mathsf{d}(\gamma_t, \gamma_s) = |t-s|\,\mathsf{d}(\gamma_0, \gamma_1)$ for every $t, s \in [0,1]$.
ii) $\implies$ iii) Observe that $\mathsf{d}(\gamma_t, \gamma_s) = (t-s)\,\mathsf{d}(\gamma_0, \gamma_1) = \int_s^t \mathsf{d}(\gamma_0, \gamma_1)\,\mathrm{d}t$ holds for every $t, s \in [0,1]$ with $s < t$, whence the curve γ is AC. Moreover, $|\dot\gamma_t| = \lim_{h \to 0} \mathsf{d}(\gamma_{t+h}, \gamma_t)/|h| = \mathsf{d}(\gamma_0, \gamma_1)$ holds for a.e. $t \in [0,1]$, thus accordingly $\int_0^1 |\dot\gamma_t|\,\mathrm{d}t = \mathsf{d}(\gamma_0, \gamma_1)$.
iii) $\implies$ iv) Clearly $|\dot\gamma_t| = \mathsf{d}(\gamma_0, \gamma_1)$ for a.e. $t \in [0,1]$, hence $\mathrm{KE}(\gamma) = \int_0^1 |\dot\gamma_t|^2\,\mathrm{d}t = \mathsf{d}(\gamma_0, \gamma_1)^2$.
iv) $\implies$ i) Notice that the function $(0,+\infty)^2 \ni (a,b) \mapsto a^2/b$ is convex and 1-homogeneous, therefore subadditive. Also, γ is AC since $\mathrm{KE}(\gamma) < \infty$. Then for all $t, s \in (0,1)$ with $s < t$ one has

$$\begin{aligned}
\mathsf{d}(\gamma_0, \gamma_1)^2 &= \int_0^s |\dot\gamma_r|^2\,\mathrm{d}r + \int_s^t |\dot\gamma_r|^2\,\mathrm{d}r + \int_t^1 |\dot\gamma_r|^2\,\mathrm{d}r \\
&\ge \frac{1}{s}\left(\int_0^s |\dot\gamma_r|\,\mathrm{d}r\right)^2 + \frac{1}{t-s}\left(\int_s^t |\dot\gamma_r|\,\mathrm{d}r\right)^2 + \frac{1}{1-t}\left(\int_t^1 |\dot\gamma_r|\,\mathrm{d}r\right)^2 \\
&\ge \frac{\mathsf{d}(\gamma_0, \gamma_s)^2}{s} + \frac{\mathsf{d}(\gamma_s, \gamma_t)^2}{t-s} + \frac{\mathsf{d}(\gamma_t, \gamma_1)^2}{1-t} \\
&\ge \frac{\big[\mathsf{d}(\gamma_0, \gamma_s) + \mathsf{d}(\gamma_s, \gamma_t) + \mathsf{d}(\gamma_t, \gamma_1)\big]^2}{s + (t-s) + (1-t)} \ge \mathsf{d}(\gamma_0, \gamma_1)^2,
\end{aligned}$$

where the last line follows from the subadditivity of the function $(0,+\infty)^2 \ni (a,b) \mapsto a^2/b$. Hence all inequalities are actually equalities, which forces $\mathsf{d}(\gamma_t, \gamma_s) = (t-s)\,\mathsf{d}(\gamma_0, \gamma_1)$. □

Let us define

$$\mathrm{Geo}(\mathrm{X}) := \big\{\gamma \in C([0,1], \mathrm{X}) \,:\, \gamma \text{ is a geodesic}\big\}. \tag{3.11}$$

Since uniform limits of geodesic curves are geodesic, we have that $\mathrm{Geo}(\mathrm{X})$ is $\underline{\mathsf{d}}$-closed.

Definition 3.11 (Geodesic space) *We say (X, d) is a* geodesic space *provided for any pair of points $x, y \in \mathrm{X}$ there exists a curve $\gamma \in \mathrm{Geo}(\mathrm{X})$ such that $\gamma_0 = x$ and $\gamma_1 = y$.*

Proposition 3.12 (Kuratowski embedding) *Let (X, d) be complete and separable. Then there exists a complete, separable and geodesic metric space $(\tilde{\mathrm{X}}, \tilde{\mathsf{d}})$ such that X is isometrically embedded into $\tilde{\mathrm{X}}$.*

Proof. Fix $(x_n)_n \subseteq \mathrm{X}$ dense. Let us define the map $\iota : \mathrm{X} \to \ell^\infty$ as follows:

$$\iota(x) := \big(\mathsf{d}(x, x_n) - \mathsf{d}(x_0, x_n)\big)_n \qquad \text{for every } x \in \mathrm{X}.$$

Since $\big|\mathsf{d}(x, x_n) - \mathsf{d}(x_0, x_n)\big| \le \mathsf{d}(x, x_0)$ for any $n \in \mathbb{N}$, we see that $\iota(x)$ actually belongs to ℓ^∞ for every $x \in \mathrm{X}$. By arguing as in the proof of Theorem 3.6, precisely when we showed (3.6), we deduce from the density of $(x_n)_n$ in X that

$$\big\|\iota(x) - \iota(y)\big\|_{\ell^\infty} = \sup_{n \in \mathbb{N}} \big|\mathsf{d}(x, x_n) - \mathsf{d}(y, x_n)\big| = \mathsf{d}(x, y) \qquad \text{holds for every } x, y \in \mathrm{X},$$

which proves that ι is an isometry. The Banach space ℓ^∞ is clearly geodesic, but it is not separable, so that we cannot just take $\tilde{\mathrm{X}} = \ell^\infty$. We thus proceed as follows: call $\mathrm{X}_0 := \iota(\mathrm{X})$ and recursively define $\mathrm{X}_{n+1} := \big\{\lambda\, x + (1-\lambda)\, y \;:\; \lambda \in [0,1],\, x, y \in \mathrm{X}_n\big\}$ for every $n \in \mathbb{N}$. Finally, let us denote $\tilde{\mathrm{X}} := \mathrm{cl}_{\ell^\infty} \bigcup_n \mathrm{X}_n$, which is the closed convex hull of X_0. Note that X is separable, so that X_0 and accordingly $\tilde{\mathrm{X}}$ are separable, and that $\iota : \mathrm{X} \to \tilde{\mathrm{X}}$ is an isometry. Since $\tilde{\mathrm{X}}$ is also complete and geodesic, we get the thesis. □

4 Lesson [18/10/2017]

Consider two metric spaces $(\mathrm{X}, \mathsf{d}_\mathrm{X}), (\mathrm{Y}, \mathsf{d}_\mathrm{Y})$ and a Borel map $T : \mathrm{X} \to \mathrm{Y}$. Given a Borel measure $\mu \geq 0$ on X, we define the *pushforward measure* $T_*\mu$ as

$$T_*\mu(E) := \mu\big(T^{-1}(E)\big) \quad \text{for every } E \subseteq \mathrm{X} \text{ Borel.} \tag{4.1}$$

It can be readily checked that $T_*\mu$ is a Borel measure on Y.

Remark 4.1 In general, if μ is a Radon measure then $T_*\mu$ is not necessarily a Radon measure. However, if μ is a finite measure then $T_*\mu$ is a Radon measure by Corollary 2.4. ■

Example 4.2 Consider the projection map $\mathbb{R}^2 \ni (x,y) \mapsto \pi^1(x,y) := x \in \mathbb{R}$. Given any Borel set $E \subseteq \mathbb{R}$, it holds that $\pi^1_*\mathcal{L}^2(E) = 0$ if $\mathcal{L}^1(E) = 0$ and $\pi^1_*\mathcal{L}^2(E) = +\infty$ if $\mathcal{L}^1(E) > 0$.

Proposition 4.3 *Let $\nu \geq 0$ be a Borel measure on* Y*. Then $\nu = T_*\mu$ if and only if*

$$\int f \,\mathrm{d}\nu = \int f \circ T \,\mathrm{d}\mu \quad \textit{for every } f : \mathrm{X} \to [0,+\infty] \textit{ Borel.} \tag{4.2}$$

We shall call (4.2) *the* change-of-variable formula.

Proof. Given $E \subseteq \mathrm{Y}$ Borel and supposing the validity of (4.2), we have that

$$\nu(E) = \int \chi_E \,\mathrm{d}\nu = \int \chi_E \circ T \,\mathrm{d}\mu = \int \chi_{T^{-1}(E)} \,\mathrm{d}\mu = \mu\big(T^{-1}(E)\big) = T_*\mu(E),$$

proving sufficiency. On the other hand, by Cavalieri's principle we see that

$$\int f \,\mathrm{d}T_*\mu = \int_0^{+\infty} T_*\mu\big(\{f \geq t\}\big) \,\mathrm{d}t = \int_0^{+\infty} \mu\big(\{f \circ T \geq t\}\big) \,\mathrm{d}t = \int f \circ T \,\mathrm{d}\mu$$

is satisfied for any Borel map $f : \mathrm{X} \to [0,+\infty]$, granting also necessity. □

Remark 4.4 Observe that

$$\begin{aligned} T = \tilde{T} \quad \mu\text{-a.e.} \quad &\Longrightarrow \quad T_*\mu = \tilde{T}_*\mu, \\ f = \tilde{f} \quad (T_*\mu)\text{-a.e.} \quad &\Longrightarrow \quad f \circ T = \tilde{f} \circ T \quad \mu\text{-a.e.}. \end{aligned} \tag{4.3}$$

Moreover, if $\nu \geq 0$ is a Borel measure on Y satisfying $T_*\mu \leq C\nu$ for some $C > 0$ and $p \in [1, \infty]$, then the operator $L^p(\nu) \ni f \mapsto f \circ T \in L^p(\mu)$ is well-defined, linear and continuous. Indeed, we have for any $f \in L^p(\nu)$ that

$$\int |f \circ T|^p \,\mathrm{d}\mu = \int |f|^p \circ T \,\mathrm{d}\mu \overset{(4.2)}{=} \int |f|^p \,\mathrm{d}T_*\mu \leq C \int |f|^p \,\mathrm{d}\nu.$$

In particular, the operator $L^p(T_*\mu) \ni f \mapsto f \circ T \in L^p(\mu)$ is an isometry. ■

Remark 4.5 Consider $f \in C^1(\mathbb{R}^n)$ and $G \in C(\mathbb{R}^n)$. Then $G \geq |\nabla f|$ if and only if

$$\big|f(\gamma_1) - f(\gamma_0)\big| \leq \int_0^1 G(\gamma_t)|\gamma_t'| \,\mathrm{d}t \quad \text{for every } \gamma \in C^1([0,1], \mathbb{R}^n). \tag{4.4}$$

This means that the map $|\nabla f|$ can be characterised, in a purely variational way, as the least continuous function $G : \mathbb{R}^n \to \mathbb{R}$ for which (4.4) is satisfied. ■

For every $t \in [0, 1]$, we define the *evaluation map* at time t as

$$\begin{aligned} \mathrm{e}_t : C([0,1], \mathrm{X}) &\longrightarrow \mathrm{X}, \\ \gamma &\longmapsto \gamma_t. \end{aligned} \tag{4.5}$$

It is clear that each function e_t is 1-Lipschitz.

Definition 4.6 (Test plan) *A measure $\boldsymbol{\pi} \in \mathscr{P}\big(C([0,1], \mathrm{X})\big)$ is said to be a* test plan *on* X *provided the following two properties are satisfied:*

i) *There exists a constant $C > 0$ such that $(\mathrm{e}_t)_*\boldsymbol{\pi} \leq C\mathfrak{m}$ for every $t \in [0, 1]$.*

ii) *It holds that $\int \mathrm{KE}(\gamma) \,\mathrm{d}\boldsymbol{\pi}(\gamma) = \int_0^1 \int |\dot{\gamma}_t|^2 \,\mathrm{d}\boldsymbol{\pi}(\gamma) \,\mathrm{d}t < +\infty$.*

The least constant $C > 0$ that can be chosen in i) *is called* compression constant *of $\boldsymbol{\pi}$ and is denoted by* $\mathrm{Comp}(\boldsymbol{\pi})$.

It follows from ii) that test plans must be concentrated on absolutely continuous curves.

Definition 4.7 (Sobolev class) *The* Sobolev class $\mathrm{S}^2(\mathrm{X})$ *is defined as the space of all Borel functions $f : \mathrm{X} \to \mathbb{R}$ that satisfy the following property: there exists a function $G \in L^2(\mathfrak{m})$ with $G \geq 0$ such that*

$$\int \big|f(\gamma_1) - f(\gamma_0)\big| \,\mathrm{d}\boldsymbol{\pi}(\gamma) \leq \int_0^1 \int G(\gamma_t)|\dot{\gamma}_t| \,\mathrm{d}\boldsymbol{\pi}(\gamma) \,\mathrm{d}t \quad \textit{for every test plan } \boldsymbol{\pi} \textit{ on } \mathrm{X}. \tag{4.6}$$

Any such G is said to be a weak upper gradient *for f.*

Remark 4.8 We claim that

$$f \circ \mathrm{e}_1 - f \circ \mathrm{e}_0 \in L^1(\boldsymbol{\pi}) \quad \text{for every } f \in \mathrm{S}^2(\mathrm{X}). \tag{4.7}$$

In order to prove, it suffices to notice that Hölder inequality gives

$$\left(\int_0^1\int G(\gamma_t)|\dot\gamma_t|\,\boldsymbol{\pi}(\gamma)\,\mathrm{d}t\right)^2 \le \left(\int_0^1\int G^2\circ \mathrm{e}_t\,\mathrm{d}\boldsymbol{\pi}\,\mathrm{d}t\right)\left(\int_0^1\int |\dot\gamma_t|^2\,\boldsymbol{\pi}(\gamma)\,\mathrm{d}t\right)$$
$$\le \mathrm{Comp}(\boldsymbol{\pi})\,\|G\|^2_{L^2(\mathfrak{m})}\int_0^1\int |\dot\gamma_t|^2\,\mathrm{d}\boldsymbol{\pi}(\gamma)\,\mathrm{d}t < +\infty.$$

In particular, the map $L^2(\mathfrak{m}) \ni G \mapsto \int_0^1\int G(\gamma_t)|\dot\gamma_t|\,\mathrm{d}\boldsymbol{\pi}(\gamma)\,\mathrm{d}t$ is linear and continuous. ■

Proposition 4.9 *Let $f \in \mathrm{S}^2(\mathrm{X})$ be fixed. Then the set of all weak upper gradients of f is closed and convex in $L^2(\mathfrak{m})$. In particular, there exists a unique weak upper gradient of f having minimal $L^2(\mathfrak{m})$-norm.*

Proof. Convexity is trivial. To prove closedness, fix a sequence $(G_n)_n \subseteq L^2(\mathfrak{m})$ of weak upper gradients of f that $L^2(\mathfrak{m})$-converges to some $G \in L^2(\mathfrak{m})$. Hence Remark 4.8 grants that

$$\int \big|f(\gamma_1)-f(\gamma_0)\big|\,\mathrm{d}\boldsymbol{\pi}(\gamma) \le \int_0^1\int G_n(\gamma_t)|\dot\gamma_t|\,\mathrm{d}\boldsymbol{\pi}(\gamma)\,\mathrm{d}t \xrightarrow{n} \int_0^1\int G(\gamma_t)|\dot\gamma_t|\,\mathrm{d}\boldsymbol{\pi}(\gamma)\,\mathrm{d}t,$$

proving that G is a weak upper gradient of f. Hence the set of weak upper gradients of f is closed. Since $L^2(\mathfrak{m})$ is Hilbert, even the last statement follows. □

Definition 4.10 (Minimal weak upper gradient) *Let $f \in \mathrm{S}^2(\mathrm{X})$. Then the unique weak upper gradient of f having minimal norm is called* minimal weak upper gradient *of f and is denoted by $|Df| \in L^2(\mathfrak{m})$.*

Proposition 4.11 *Let the sequence $(f_n)_n \subseteq \mathrm{S}^2(\mathrm{X})$ satisfy $f_n(x) \to f(x)$ for a.e. $x \in \mathrm{X}$, for some Borel map $f : \mathrm{X} \to \mathbb{R}$. Let $G_n \in L^2(\mathfrak{m})$ be a weak upper gradient of f_n for every $n \in \mathbb{N}$. Suppose that $G_n \rightharpoonup G$ weakly in $L^2(\mathfrak{m})$, for some $G \in L^2(\mathfrak{m})$. Then $f \in \mathrm{S}^2(\mathrm{X})$ and G is a weak upper gradient of f.*

Proof. First of all, it holds that $f_n(\gamma_1) - f_n(\gamma_0) \overset{n}{\to} f(\gamma_1) - f(\gamma_0)$ for $\boldsymbol{\pi}$-a.e. γ. Moreover, the map sending $H \in L^2(\mathfrak{m})$ to $\int_0^1\int H(\gamma_t)|\dot\gamma_t|\,\mathrm{d}\boldsymbol{\pi}(\gamma)\,\mathrm{d}t$ is strongly continuous and linear by Remark 4.8, thus it is also weakly continuous. Hence Fatou's lemma yields

$$\int \big|f(\gamma_1)-f(\gamma_0)\big|\,\mathrm{d}\boldsymbol{\pi}(\gamma) \le \varliminf_{n\to\infty}\int \big|f_n(\gamma_1)-f(\gamma_0)\big|\,\mathrm{d}\boldsymbol{\pi}(\gamma) \le \lim_{n\to\infty}\int_0^1\int G_n(\gamma_t)|\dot\gamma_t|\,\mathrm{d}\boldsymbol{\pi}(\gamma)\,\mathrm{d}t$$
$$= \int_0^1\int G(\gamma_t)|\dot\gamma_t|\,\mathrm{d}\boldsymbol{\pi}(\gamma)\,\mathrm{d}t,$$

which shows that $f \in \mathrm{S}^2(\mathrm{X})$ and that G is a weak upper gradient for f. □

Example 4.12 Let us fix a measure $\mu \in \mathscr{P}(\mathrm{X})$ with $\mu \le C\mathfrak{m}$ for some $C > 0$. Let us denote by $\mathsf{Const} : \mathrm{X} \to C([0,1],\mathrm{X})$ the function sending any point $x \in \mathrm{X}$ to the curve identically equal to x. Then $\mathsf{Const}_*\mu$ turns out to be a test plan on X.

Exercise 4.13 Given a metric space (X, d) and $\alpha \in (0,1)$, we define the distance d_α on X as

$$\mathsf{d}_\alpha(x,y) := \mathsf{d}(x,y)^\alpha \quad \text{for every } x,y \in \mathrm{X}.$$

Prove that the metric space $(\mathrm{X}, \mathsf{d}_\alpha)$, which is called the *snowflaking* of (X, d), has the following property: if a curve γ is d_α-absolutely continuous, then it is constant.

Now consider any Borel measure $\mathfrak{m}$ on (X, d). Since d and d_α induce the same topology on X, we have that $\mathfrak{m}$ is also a Borel measure on $(\mathrm{X}, \mathsf{d}_\alpha)$. Prove that any Borel map on X belongs to $\mathrm{S}^2(\mathrm{X}, \mathsf{d}_\alpha, \mathfrak{m})$ and has null minimal weak upper gradient.

Definition 4.14 (Sobolev space) *We define the* Sobolev space $W^{1,2}(\mathrm{X})$ *associated to the metric measure space* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *as* $W^{1,2}(\mathrm{X}) := L^2(\mathfrak{m}) \cap \mathrm{S}^2(\mathrm{X})$*. Moreover, we define*

$$\|f\|_{W^{1,2}(\mathrm{X})} := \Big(\|f\|^2_{L^2(\mathfrak{m})} + \big\||Df|\big\|^2_{L^2(\mathfrak{m})}\Big)^2 \quad \textit{for every } f \in W^{1,2}(\mathrm{X}). \tag{4.8}$$

Remark 4.15 It is trivial to check that

$$\begin{aligned} \big|D(\lambda f)\big| &= |\lambda||Df| \quad \text{for every } f \in \mathrm{S}^2(\mathrm{X}) \text{ and } \lambda \in \mathbb{R}, \\ \big|D(f+g)\big| &\leq |Df| + |Dg| \quad \text{for every } f,g \in \mathrm{S}^2(\mathrm{X}). \end{aligned} \tag{4.9}$$

In particular, $\mathrm{S}^2(\mathrm{X})$ is a vector space, so accordingly $W^{1,2}(\mathrm{X})$ is a vector space as well. ■

Theorem 4.16 *The space* $\big(W^{1,2}(\mathrm{X}), \|\cdot\|_{W^{1,2}(\mathrm{X})}\big)$ *is a Banach space.*

Proof. First of all, we claim that $\mathrm{S}^2(\mathrm{X}) \ni f \mapsto \big\||Df|\big\|_{L^2(\mathfrak{m})} \in \mathbb{R}$ is a seminorm: it follows by taking the $L^2(\mathfrak{m})$-norm in (4.9). Then also $\|\cdot\|_{W^{1,2}(\mathrm{X})}$ is a seminorm. Actually, it is a norm because $\|f\|_{W^{1,2}(\mathrm{X})} = 0$ implies $\|f\|_{L^2(\mathfrak{m})} = 0$ and accordingly $f = 0$. It thus remains to show that $W^{1,2}(\mathrm{X})$ is complete. To this aim, fix a Cauchy sequence $(f_n)_n \subseteq W^{1,2}(\mathrm{X})$. In particular, such sequence is $L^2(\mathfrak{m})$-Cauchy, so that it has an $L^2(\mathfrak{m})$-limit f. Moreover, the sequence $\big(|Df_n|\big)_n$ is bounded in $L^2(\mathfrak{m})$. Hence there exists a subsequence $(f_{n_k})_k$ such that

$$\begin{aligned} |Df_{n_k}| &\rightharpoonup G \quad \text{weakly in } L^2(\mathfrak{m}), \text{ for some } G \in L^2(\mathfrak{m}), \\ f_{n_k}(x) &\xrightarrow{k} f(x) \quad \text{for } \mathfrak{m}\text{-a.e. } x \in \mathrm{X}. \end{aligned} \tag{4.10}$$

Then Proposition 4.11 grants that $f \in W^{1,2}(\mathrm{X})$ and that G is a weak upper gradient for f. Finally, with a similar argument we get $\big\||D(f_{n_k} - f)|\big\|_{L^2(\mathfrak{m})} \leq \underline{\lim}_m \big\||D(f_{n_k} - f_{n_m})|\big\|_{L^2(\mathfrak{m})}$ for every $k \in \mathbb{N}$. By recalling that $(f_n)_n$ is $W^{1,2}(\mathrm{X})$-Cauchy, we thus conclude that

$$\varlimsup_{k\to\infty} \big\||D(f_{n_k} - f)|\big\|_{L^2(\mathfrak{m})} \leq \varlimsup_{k\to\infty} \varliminf_{m\to\infty} \big\||D(f_{n_k} - f_{n_m})|\big\|_{L^2(\mathfrak{m})} = 0,$$

proving that $f_{n_k} \to f$ in $W^{1,2}(\mathrm{X})$, which in turn grants that $f_n \to f$ in $W^{1,2}(\mathrm{X})$. □

Remark 4.17 In general, $W^{1,2}(\mathrm{X})$ is not a Hilbert space. For instance, $W^{1,2}(\mathbb{R}^n, \mathsf{d}, \mathcal{L}^n)$ is not Hilbert for any distance d induced by a norm not coming from a scalar product. ■

Theorem 4.18 *Let $f : \mathrm{X} \to \mathbb{R}$ be a Borel map. Let $G \in L^2(\mathfrak{m})$ satisfy $G \geq 0$. Then the following are equivalent:*

i) *$f \in \mathrm{S}^2(\mathrm{X})$ and G is a weak upper gradient of f.*

ii) *For every test plan $\boldsymbol{\pi}$ on X, we have that the map $t \mapsto f \circ \mathrm{e}_t - f \circ \mathrm{e}_0 \in L^1(\boldsymbol{\pi})$ is AC. For a.e. $t \in [0,1]$, there exists the strong $L^1(\boldsymbol{\pi})$-limit of $(f \circ \mathrm{e}_{t+h} - f \circ \mathrm{e}_t)/h$ as $h \to 0$. Such limit, denoted by $\mathsf{Der}_{\boldsymbol{\pi}}(f)_t \in L^1(\boldsymbol{\pi})$, satisfies*

$$\big|\mathsf{Der}_{\boldsymbol{\pi}}(f)_t\big|(\gamma) \leq G(\gamma_t)|\dot{\gamma}_t| \quad \textit{for } \boldsymbol{\pi}\textit{-a.e. } \gamma \textit{ and a.e. } t \in [0,1]. \tag{4.11}$$

iii) *For every test plan $\boldsymbol{\pi}$, we have for $\boldsymbol{\pi}$-a.e. γ that $f \circ \gamma$ belongs to $W^{1,1}(0,1)$ and that the inequality $\big|\partial_t(f \circ \gamma_t)\big| \leq G(\gamma_t)|\dot{\gamma}_t|$ holds for a.e. $t \in [0,1]$.*

If the above hold, then $\mathsf{Der}_{\boldsymbol{\pi}}(f)_t(\gamma) = \partial_t(f \circ \gamma_t)$ for $\boldsymbol{\pi}$-a.e. γ and a.e. $t \in [0,1]$.

Remark 4.19 One can deduce from ii) of Theorem 4.18 that

$$G_1, G_2 \text{ weak upper gradients for } f \quad \Longrightarrow \quad \min\{G_1, G_2\} \text{ weak upper gradient for } f. \tag{4.12}$$

In particular, we have that the minimal weak upper gradient is minimal also in the $\mathfrak{m}$-a.e. sense, i.e. $|Df| \leq G$ holds $\mathfrak{m}$-a.e. for every weak upper gradient G of f. ■

5 Lesson [23/10/2017]

Remark 5.1 In giving Definition 4.7 we implicitly used the fact that

$$C([0,1],\mathrm{X}) \times [0,1] \ni (\gamma,t) \longmapsto G(\gamma_t)|\dot{\gamma}_t| \quad \text{is Borel.} \tag{5.1}$$

The map $\mathrm{e} : C([0,1],\mathrm{X}) \times [0,1] \to \mathrm{X}$ sending (γ,t) to γ_t can be easily seen to be continuous, whence $G \circ \mathrm{e}$ is Borel. Moreover, define the map $\mathsf{ms} : C([0,1],X) \times [0,1] \longrightarrow [0,+\infty]$ as

$$\mathsf{ms}(\gamma,t) := \begin{cases} |\dot{\gamma}_t| = \lim_{h \to 0} \mathsf{d}(\gamma_{t+h},\gamma_t)/|h| & \text{if such limit exists finite,} \\ +\infty & \text{otherwise.} \end{cases}$$

We claim that ms is Borel. To prove it, consider an enumeration $(r_n)_n$ of $\mathbb{Q} \cap (0,+\infty)$. Given any $\varepsilon, h > 0$ and $n \in \mathbb{N}$, we define the Borel sets $A(\varepsilon,n,h)$ and $B(\varepsilon,n)$ as follows:

$$A(\varepsilon,n,h) := \left\{(\gamma,t) \;:\; \left|\frac{\mathsf{d}(\gamma_{t+h},\gamma_t)}{|h|} - r_n\right| < \varepsilon\right\}, \quad B(\varepsilon,n) := \bigcup_{0<\delta\in\mathbb{Q}} \bigcap_{h\in(0,\delta)\cap\mathbb{Q}} A(\varepsilon,n,h).$$

Hence $\lim_{h\to 0} \mathsf{d}(\gamma_{t+h},\gamma_t)/|h|$ exists finite if and only if $(\gamma,t) \in \bigcap_{j\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} B(2^{-j},n)$. Now let us call $C(j,n) := B(2^{-j},n) \setminus \bigcup_{i<n} B(2^{-j},i)$ for every $j,n \in \mathbb{N}$. Then the map f_j, defined as

$$f_j(\gamma,t) := \begin{cases} r_n & \text{if } (\gamma,t) \in C(j,n) \text{ for some } n \in \mathbb{N}, \\ +\infty & \text{if } (\gamma,t) \notin \bigcup_n C(j,n), \end{cases}$$

is Borel by construction. Given that $f_j(\gamma,t) \overset{j}{\to} \mathsf{ms}(\gamma,t)$ for every (γ,t), we thus deduce that the function ms is Borel. Since the map in (5.1) is nothing but $G \circ \mathrm{e}\, \mathsf{ms}$, we finally conclude that (5.1) is satisfied. ■

Proposition 5.2 *Let $(f_n)_n \subseteq \mathrm{S}^2(\mathrm{X})$ be given. Suppose that there exists $f : \mathrm{X} \to \mathbb{R}$ Borel such that $f(x) = \lim_n f_n(x)$ for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$. Then $\big\||Df|\big\|_{L^2(\mathfrak{m})} \leq \varliminf_n \big\||Df_n|\big\|_{L^2(\mathfrak{m})}$.*
In particular, if a sequence $(g_n)_n \subseteq W^{1,2}(\mathrm{X})$ is $L^2(\mathfrak{m})$-converging to some limit $g \in L^2(\mathfrak{m})$, then it holds that $\big\||Dg|\big\|_{L^2(\mathfrak{m})} \leq \varliminf_n \big\||Dg_n|\big\|_{L^2(\mathfrak{m})}$.

Proof. The case $\varliminf_n \big\||Df_n|\big\|_{L^2(\mathfrak{m})} = +\infty$ is trivial, then assume that such liminf is finite. Up to subsequence, we can also assume that such liminf is actually a limit. This grants that the sequence $\big(|Df_n|\big)_n$ is bounded in $L^2(\mathfrak{m})$, thus (up to subsequence) we have that $|Df_n| \rightharpoonup G$ weakly in $L^2(\mathfrak{m})$ for some $G \in L^2(\mathfrak{m})$. Hence Proposition 4.11 grants that $f \in \mathrm{S}^2(\mathrm{X})$ and G is a weak upper gradient for f, so that $\big\||Df|\big\|_{L^2(\mathfrak{m})} \leq \|G\|_{L^2(\mathfrak{m})} \leq \varliminf_n \big\||Df_n|\big\|_{L^2(\mathfrak{m})}$.

For the last assertion, first take a subsequence such that $\varliminf_n \big\||Dg_n|\big\|_{L^2(\mathfrak{m})}$ is actually a limit and then note that there is a further subsequence $(g_{n_k})_k$ such that $g(x) = \lim_k g_{n_k}(x)$ holds for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$. To conclude, apply the first part of the statement. □

Example 5.3 Suppose to have a Borel map $F : \mathrm{X} \times [0,1] \to \mathrm{X}$, called *flow*, with the following properties: there exist two constants $L, C > 0$ such that

$$\begin{aligned} F_\cdot(x) : t \mapsto F_t(x) \quad & \text{is } L\text{-Lipschitz for every } x \in \mathrm{X}, \\ (F_t)_*\mathfrak{m} \leq C\mathfrak{m} \quad & \text{for every } t \in [0,1]. \end{aligned} \tag{5.2}$$

The second requirement means, in a sense, that the mass is well-distributed by the flow F.

Now consider any measure $\mu \in \mathscr{P}(\mathrm{X})$ such that $\mu \leq c\,\mathfrak{m}$ for some $c > 0$. Then

$$\boldsymbol{\pi} := (F_\cdot)_*\mu \quad \text{is a test plan on X.} \tag{5.3}$$

Its verification is straightforward: $(\mathrm{e}_t)_*\boldsymbol{\pi} = (\mathrm{e}_t)_*(F_\cdot)_*\mu = (F_t)_*\mu \leq c\,(F_t)_*\mathfrak{m} \leq c\,C\mathfrak{m}$ shows the first property of test plans, while the fact that $\big|\dot{F}_t(x)\big| \leq L$ holds for every $x \in \mathrm{X}$ and almost every $t \in [0,1]$ grants the second one. Therefore (5.3) is proved. ■

Proposition 5.4 *Let $\boldsymbol{\pi}$ be a test plan on X and $p \in [1,\infty)$. Then for every $f \in L^p(\mathfrak{m})$ the map $[0,1] \ni t \mapsto f \circ \mathrm{e}_t \in L^p(\boldsymbol{\pi})$ is continuous.*

Proof. First of all, one has that $\int |f \circ \mathrm{e}_t|^p \,\mathrm{d}\boldsymbol{\pi} \leq \mathrm{Comp}(\boldsymbol{\pi}) \int |f|^p \,\mathrm{d}\mathfrak{m}$ for every $f \in L^p(\mathfrak{m})$. Given any $g \in C_b(\mathrm{X}) \cap L^p(\mathfrak{m})$, it holds that $\big|g(\gamma_s) - g(\gamma_t)\big|^p \to 0$ as $s \to t$ for every $\gamma \in C([0,1],\mathrm{X})$ and $|g \circ \mathrm{e}_s - g \circ \mathrm{e}_t|^p \leq 2\,\|g\|^p_{C_b(\mathrm{X})} \in L^\infty(\boldsymbol{\pi})$, so that $\lim_{s\to t} \int \big|g \circ \mathrm{e}_s - g \circ \mathrm{e}_t\big|^p \,\mathrm{d}\boldsymbol{\pi} = 0$ by dominated convergence theorem. This guarantees that

$$\begin{aligned} \varlimsup_{s\to t} \|f \circ \mathrm{e}_s - f \circ \mathrm{e}_t\|_{L^p(\boldsymbol{\pi})} &\leq \varlimsup_{s\to t} \Big[\|f \circ \mathrm{e}_s - g \circ \mathrm{e}_s\|_{L^p(\boldsymbol{\pi})} + \|g \circ \mathrm{e}_t - f \circ \mathrm{e}_t\|_{L^p(\boldsymbol{\pi})}\Big] \\ &\leq 2\,\mathrm{Comp}(\boldsymbol{\pi})^{1/p}\,\|f - g\|_{L^p(\mathfrak{m})}, \end{aligned}$$

whence $\|f \circ \mathrm{e}_s - f \circ \mathrm{e}_t\|_{L^p(\boldsymbol{\pi})} \to 0$ as $s \to t$ by density of $C_b(\mathrm{X}) \cap L^p(\mathfrak{m})$ in $L^p(\mathfrak{m})$, which can be proved by suitably adapting the proof of Proposition 2.5. □

Let $t, s \in [0,1]$ be fixed. Then we define the map $\mathsf{Restr}_t^s : C([0,1],\mathrm{X}) \to C([0,1],\mathrm{X})$ as

$$\mathsf{Restr}_t^s(\gamma)_r := \gamma_{(1-r)t+rs} \qquad \text{for every } \gamma \in C([0,1],\mathrm{X}) \text{ and } r \in [0,1]. \tag{5.4}$$

Exercise 5.5 Prove that the map Restr_t^s is continuous.

Lemma 5.6 *Let $\boldsymbol{\pi}$ be a test plan on* X. *Then*

i) *for any Borel set $\Gamma \subseteq C([0,1],\mathrm{X})$ that satisfies $\boldsymbol{\pi}(\Gamma) > 0$, it holds that $\boldsymbol{\pi}(\Gamma)^{-1}\,\boldsymbol{\pi}_{|\Gamma}$ is a test plan on* X*,*

ii) *the measure $(\mathsf{Restr}_t^s)_*\boldsymbol{\pi}$ is a test plan on* X.

Proof. In order to prove i), just observe that

$$\begin{aligned}
&(\mathrm{e}_t)_*\big(\boldsymbol{\pi}(\Gamma)^{-1}\,\boldsymbol{\pi}_{|\Gamma}\big) \leq \boldsymbol{\pi}(\Gamma)^{-1}\,(\mathrm{e}_t)_*\boldsymbol{\pi} \leq \mathrm{Comp}(\boldsymbol{\pi})\,\boldsymbol{\pi}(\Gamma)^{-1}\,\mathfrak{m},\\
&\int_0^1\!\!\int |\dot\gamma_t|^2\,\mathrm{d}\big(\boldsymbol{\pi}(\Gamma)^{-1}\,\boldsymbol{\pi}_{|\Gamma}\big)(\gamma)\,\mathrm{d}t = \boldsymbol{\pi}(\Gamma)^{-1}\int_0^1\!\!\int_\Gamma |\dot\gamma_t|^2\,\mathrm{d}\boldsymbol{\pi}(\gamma)\,\mathrm{d}t < +\infty.
\end{aligned}$$

To prove ii), notice that if $\gamma \in C([0,1],\mathrm{X})$ is absolutely continuous, then $\sigma := \mathsf{Restr}_t^s(\gamma)$ is absolutely continuous as well and satisfies $|\dot\sigma_r| = |s-t|\,|\dot\gamma_{(1-r)t+rs}|$ for a.e. $r \in [0,1]$. Hence

$$\begin{aligned}
&(\mathrm{e}_r)_*(\mathsf{Restr}_t^s)_*\boldsymbol{\pi} = (\mathrm{e}_r \circ \mathsf{Restr}_t^s)_*\boldsymbol{\pi} = (\mathrm{e}_{(1-r)t+rs})_*\boldsymbol{\pi} \leq \mathrm{Comp}(\boldsymbol{\pi})\,\mathfrak{m},\\
&\int_0^1\!\!\int |\dot\sigma_r|^2\,\mathrm{d}\big((\mathsf{Restr}_t^s)_*\boldsymbol{\pi}\big)(\sigma)\,\mathrm{d}r \leq |s-t|\int_0^1\!\!\int |\dot\gamma_r|^2\,\mathrm{d}\boldsymbol{\pi}(\gamma)\,\mathrm{d}r < +\infty,
\end{aligned}$$

which concludes the proof of the statement. □

Proposition 5.7 *Let $f \in \mathrm{S}^2(\mathrm{X})$ be given. Consider a weak upper gradient $G \in L^2(\mathfrak{m})$ of f. Then for every test plan $\boldsymbol{\pi}$ on* X *and for every $t, s \in [0,1]$ with $s < t$ it holds that*

$$\big|f(\gamma_t) - f(\gamma_s)\big| \leq \int_s^t G(\gamma_r)|\dot\gamma_r|\,\mathrm{d}r \qquad \textit{for } \boldsymbol{\pi}\textit{-a.e. } \gamma \in C([0,1],\mathrm{X}). \tag{5.5}$$

Proof. We argue by contradiction: suppose the existence of $t, s \in [0,1]$ with $s < t$ and of a Borel set $\Gamma \subseteq C([0,1],\mathrm{X})$ with $\boldsymbol{\pi}(\Gamma) > 0$ such that $\big|f(\gamma_t) - f(\gamma_s)\big| > \int_s^t G(\gamma_r)|\dot\gamma_r|\,\mathrm{d}r$ holds for every $\gamma \in \Gamma$. Lemma 5.6 grants that the measure $\tilde{\boldsymbol{\pi}} := (\mathsf{Restr}_s^t)_*\big(\boldsymbol{\pi}(\Gamma)^{-1}\,\boldsymbol{\pi}_{|\Gamma}\big)$ is a test plan on X, thus accordingly

$$\begin{aligned}
\boldsymbol{\pi}(\Gamma)^{-1}\int_\Gamma \big|f(\gamma_t) - f(\gamma_s)\big|\,\mathrm{d}\boldsymbol{\pi}(\gamma) &= \int \big|f(\sigma_1) - f(\sigma_0)\big|\,\mathrm{d}\tilde{\boldsymbol{\pi}}(\sigma) \leq \int_0^1\!\!\int G(\sigma_r)|\dot\sigma_r|\,\mathrm{d}\tilde{\boldsymbol{\pi}}(\sigma)\,\mathrm{d}r\\
&= \boldsymbol{\pi}(\Gamma)^{-1}\int_s^t\!\!\int_\Gamma G(\gamma_r)|\dot\gamma_r|\,\mathrm{d}\boldsymbol{\pi}(\gamma)\,\mathrm{d}r,
\end{aligned}$$

which leads to a contradiction. Therefore the thesis is achieved. □

Fix a Banach space $\mathbb{B}$ and a metric measure space $(\mathrm{X},\mathsf{d},\mu)$ with $\mu\in\mathscr{P}(\mathrm{X})$. A map $f:\mathrm{X}\to\mathbb{B}$ is said to be *simple* provided it can be written as $f=\sum_{i=1}^n\chi_{E_i}\,v_i$, for some $v_1,\ldots,v_n\in\mathbb{B}$ and some Borel partition $E_1,\ldots,E_n$ of X.

Definition 5.8 (Strongly Borel) *A map $f:\mathrm{X}\to\mathbb{B}$ is said to be* strongly Borel *(resp.* strongly μ-measurable*) provided it is Borel (resp. μ-measurable) and there exists a separable subset V of $\mathbb{B}$ such that $f(x)\in V$ for μ-a.e. $x\in\mathrm{X}$. This last condition can be briefly expressed by saying that f is* essentially separably valued*.*

Lemma 5.9 *Let $f:\mathrm{X}\to\mathbb{B}$ be any given map. Then f is strongly Borel if and only if it is Borel and there exists a sequence $(f_n)_n$ of simple maps such that $\lim_n\big\|f_n(x)-f(x)\big\|_{\mathbb{B}}=0$ is satisfied for μ-a.e. $x\in\mathrm{X}$.*

Proof. Sufficiency. Choose $V_n\subseteq\mathbb{B}$ separable such that $f_n(x)\in V_n$ for μ-a.e. $x\in\mathrm{X}$. Then the set $V:=\overline{\bigcup_n V_n}$ is separable and $f(x)\in V$ for μ-a.e. $x\in\mathrm{X}$, whence f is strongly Borel.
Necessity. Assume wlog $f(x)\in V$ for every $x\in\mathrm{X}$. Choose a dense countable subset $(v_n)_n$ of V and notice that $V\subseteq\bigcup_n B_\varepsilon(v_n)$ for every $\varepsilon>0$. We define $P_\varepsilon:V\to(v_n)_n$ as follows:

$$P_\varepsilon:=\sum_{n\in\mathbb{N}}\chi_{C(\varepsilon,n)}\,v_n,\qquad\text{where }C(\varepsilon,n):=\big(V\cap B_\varepsilon(v_n)\big)\setminus\bigcup_{i<n}B_\varepsilon(v_i).\tag{5.6}$$

Let us call $f_\varepsilon:=P_\varepsilon\circ f$. Since $\big\|P_\varepsilon(v)-v\big\|_{\mathbb{B}}\le\varepsilon$ for all $v\in V$, we have that $\big\|f_\varepsilon(x)-f(x)\big\|_{\mathbb{B}}\le\varepsilon$ for all $x\in\mathrm{X}$, so that f can be pointwise approximated by maps taking countably many values. With a cut-off argument, we can then approximate f by simple maps, as required. $\square$

Given a simple map $f:\mathrm{X}\to\mathbb{B}$ and a Borel set $E\subseteq\mathrm{X}$, we define

$$\int_E f\,\mathrm{d}\mu:=\sum_{i=1}^n\mu(E_i\cap E)\,v_i\in\mathbb{B}\qquad\text{if }f=\sum_{i=1}^n\chi_{E_i}\,v_i.\tag{5.7}$$

Exercise 5.10 Show that the integral in (5.7) is well-posed, i.e. it does not depend on the particular way of writing f, and that it is linear.

Definition 5.11 (Bochner integral) *A map $f:\mathrm{X}\to\mathbb{B}$ is said to be* Bochner integrable *provided there exists a sequence $(f_n)_n$ of simple maps such that each $x\mapsto\big\|f_n(x)-f(x)\big\|_{\mathbb{B}}$ is a μ-measurable function and $\lim_n\int\|f_n-f\|_{\mathbb{B}}\,\mathrm{d}\mu=0$. In this case, we define*

$$\int_E f\,\mathrm{d}\mu:=\lim_{n\to\infty}\int_E f_n\,\mathrm{d}\mu\qquad\textit{for every }E\subseteq\mathrm{X}\textit{ Borel.}\tag{5.8}$$

Remark 5.12 It follows from the very definition that the inequality

$$\left\|\int_E f\,\mathrm{d}\mu\right\|_{\mathbb{B}}\le\int_E\|f\|_{\mathbb{B}}\,\mathrm{d}\mu\tag{5.9}$$

holds for every f simple. Now fix a Bochner integrable map f and a sequence $(f_n)_n$ of simple maps that converge to f as in Definition 5.11. Hence we have that

$$\left\|\int_E(f_n-f_m)\,\mathrm{d}\mu\right\|_{\mathbb{B}}\overset{(5.9)}{\le}\int_E\|f_n-f\|_{\mathbb{B}}\,\mathrm{d}\mu+\int_E\|f-f_m\|_{\mathbb{B}}\,\mathrm{d}\mu\xrightarrow{n,m}0,$$

proving that $\big(\int_E f_n \,\mathrm{d}\mu\big)_n$ is Cauchy in $\mathbb{B}$ and accordingly the limit in (5.8) exists. Further, take another sequence $(g_n)_n$ of simple maps converging to f in the sense of Definition 5.11. Therefore one has that

$$\left\|\int_E (f_n - g_n)\,\mathrm{d}\mu\right\|_{\mathbb{B}} \overset{(5.9)}{\leq} \int_E \|f_n - f\|_{\mathbb{B}}\,\mathrm{d}\mu + \int_E \|f - g_n\|_{\mathbb{B}}\,\mathrm{d}\mu \xrightarrow{n} 0,$$

which implies $\lim_n \int_E f_n \,\mathrm{d}\mu = \lim_n \int_E g_n \,\mathrm{d}\mu$. This grants that $\int_E f\,\mathrm{d}\mu$ is well-defined. ■

Proposition 5.13 *Let $f : \mathrm{X} \to \mathbb{B}$ be a given map. Then f is Bochner integrable if and only if it is strongly μ-measurable and $\int \|f\|_{\mathbb{B}}\,\mathrm{d}\mu < +\infty$.*

Proof. Necessity is trivial. To prove sufficiency, consider the maps P_ε defined in (5.6) and call $f_\varepsilon := P_\varepsilon \circ f$. Hence we have $\int \|f_\varepsilon - f\|_{\mathbb{B}}\,\mathrm{d}\mu \leq \varepsilon$ for all $\varepsilon > 0$. Recall that the projection maps P_ε are written in the form $\sum_{n\in\mathbb{N}} \chi_{C(\varepsilon,n)}\, v_n$, so that $f_\varepsilon = \sum_{n\in\mathbb{N}} \chi_{f^{-1}(C(\varepsilon,n))}\, v_n$. Now let us define $g_\varepsilon^k := \sum_{n\leq k} \chi_{f^{-1}(C(\varepsilon,n))}\, v_n$ for all $k \in \mathbb{N}$. Given that $\sum_{n\in\mathbb{N}} \mu\big(f^{-1}(C(\varepsilon,n))\big)\,\|v_n\|_{\mathbb{B}}$ is equal to $\int \|f_\varepsilon\|_{\mathbb{B}}\,\mathrm{d}\mu$, which is smaller than $\int \|f\|_{\mathbb{B}}\,\mathrm{d}\mu + \varepsilon$ and accordingly finite, we see that

$$\int \|g_\varepsilon^k - f_\varepsilon\|_{\mathbb{B}}\,\mathrm{d}\mu = \sum_{n=k+1}^{\infty} \mu\big(f^{-1}(C(\varepsilon,n))\big)\,\|v_n\|_{\mathbb{B}} \xrightarrow{k} 0.$$

Since the maps g_ε^k are simple, we can thus conclude by a diagonalisation argument. □

Example 5.14 Denote by $\mathcal{M}([0,1])$ the Banach space of all signed Radon measures on $[0,1]$, endowed with the total variation norm. Then the map $[0,1] \to \mathcal{M}([0,1])$, which sends $t \in [0,1]$ to $\delta_t \in \mathscr{P}([0,1])$, is not strongly Borel.

Indeed, notice that $\|\delta_t - \delta_s\|_{\mathsf{TV}} = 2$ for every $t, s \in [0,1]$ with $t \neq s$. Now suppose that there exists a Borel set $N \subseteq [0,1]$ with $\mathcal{L}^1(N) = 0$ such that $\big\{\delta_t : t \in [0,1] \setminus N\big\}$ is separable. Take a countable dense subset $(\mu_n)_n$ of such set. Hence for every $t \in [0,1] \setminus N$ we can choose an index $\underline{n}(t) \in \mathbb{N}$ such that $\|\delta_t - \mu_{\underline{n}(t)}\|_{\mathsf{TV}} < 1$. Clearly the function $\underline{n} : [0,1] \setminus N \to \mathbb{N}$ must be injective, which contradicts the fact that $[0,1] \setminus N$ is not countable.

6 Lesson [25/10/2017]

Let us define the space $L^1(\mu;\mathbb{B})$ as follows:

$$L^1(\mu;\mathbb{B}) := \big\{f : \mathrm{X} \to \mathbb{B} \text{ Bochner integrable}\big\}/(\mu\text{-a.e. equality}). \tag{6.1}$$

Then $L^1(\mu;\mathbb{B})$ is a Banach space if endowed with the norm $\|f\|_{L^1(\mu;\mathbb{B})} := \int \big\|f(x)\big\|_{\mathbb{B}}\,\mathrm{d}\mu(x)$.

Remark 6.1 Given two metric spaces X, Y and a continuous map $f : \mathrm{X} \to \mathrm{Y}$, we have that the image $f(\mathrm{X})$ is separable whenever X is separable.

Indeed, if $(x_n)_n$ is dense in X, then $\big(f(x_n)\big)_n$ is dense in $f(\mathrm{X})$ by continuity of f. ■

Proposition 6.2 *Let $E \subseteq \mathrm{X}$ be Borel. Let $\mathbb{V}$ be another Banach space. Then:*

i) *For every $f \in L^1(\mu;\mathbb{B})$, it holds that*

$$\left\| \int_E f \, \mathrm{d}\mu \right\|_{\mathbb{B}} \leq \int_E \|f\|_{\mathbb{B}} \, \mathrm{d}\mu. \tag{6.2}$$

In particular, the map $L^1(\mu;\mathbb{B}) \to \mathbb{B}$ sending f to $\int f \, \mathrm{d}\mu$ is linear and continuous.

ii) *The space $C_b(\mathrm{X}, \mathbb{B})$ is (contained and) dense in $L^1(\mu;\mathbb{B})$.*

iii) *If $\ell : \mathbb{B} \to \mathbb{V}$ is linear continuous and $f \in L^1(\mu;\mathbb{B})$, one has that $\ell \circ f \in L^1(\mu;\mathbb{V})$ and*

$$\ell\left(\int_E f \, \mathrm{d}\mu\right) = \int_E \ell \circ f \, \mathrm{d}\mu. \tag{6.3}$$

Proof. i) As already mentioned in (5.9), we have that the inequality (6.2) is satisfied whenever the map f is simple, because if $f = \sum_{i=1}^n \chi_{E_i} v_i$ then

$$\left\| \int_E f \, \mathrm{d}\mu \right\|_{\mathbb{B}} \leq \sum_{i=1}^n \left\| \int \chi_{E_i \cap E} \, v_i \, \mathrm{d}\mu \right\|_{\mathbb{B}} = \sum_{i=1}^n \mu(E_i \cap E) \, \|v_i\|_{\mathbb{B}} = \int_E \|f\|_{\mathbb{B}} \, \mathrm{d}\mu.$$

For f generic, choose a sequence $(f_n)_n$ of simple maps that converge to f in $L^1(\mu;\mathbb{B})$. Then

$$\left\| \int_E f \, \mathrm{d}\mu \right\|_{\mathbb{B}} = \lim_{n\to\infty} \left\| \int_E f_n \, \mathrm{d}\mu \right\|_{\mathbb{B}} \leq \lim_{n\to\infty} \int_E \|f_n\|_{\mathbb{B}} \, \mathrm{d}\mu = \int_E \|f\|_{\mathbb{B}} \, \mathrm{d}\mu,$$

thus proving the validity of (6.2).
ii) The elements of $C(\mathrm{X}, \mathbb{B})$, which are clearly Borel, are (essentially) separably valued by Remark 6.1, in other words they are strongly Borel. This grants that $C_b(\mathrm{X}, \mathbb{B}) \subseteq L^1(\mu;\mathbb{B})$. To prove its density, it suffices to approximate just the maps of the form $\chi_E \, v$. First choose any sequence $(C_n)_n$ of closed subsets of E with $\mu(E \setminus C_n) \searrow 0$, so that $\chi_{C_n} v \to \chi_E \, v$ with respect to the $L^1(\mu;\mathbb{B})$-norm, then for each $n \in \mathbb{N}$ notice that the maps $\big(1 - k \, \mathsf{d}(\cdot, C_n)\big)^+ v$ belong to $C_b(\mathrm{X}, \mathbb{B})$ and $L^1(\mu;\mathbb{B})$-converge to $\chi_{C_n} v$ as $k \to \infty$. So $C_b(\mathrm{X}, \mathbb{B})$ is dense in $L^1(\mu;\mathbb{B})$.
iii) In the case in which f is simple, say $f = \sum_{i=1}^n \chi_{E_i} v_i$, one has that

$$\ell\left(\int_E f \, \mathrm{d}\mu\right) = \sum_{i=1}^n \mu(E_i \cap E) \, \ell(v_i) = \int_E \ell \circ f \, \mathrm{d}\mu.$$

For a general f, choose a sequence $(f_n)_n$ of simple maps that $L^1(\mu;\mathbb{B})$-converge to f. Observe that the inequality $\int \big\|\ell(f - f_n)\big\|_{\mathbb{V}}(x) \, \mathrm{d}\mu(x) \leq \|\ell\| \int \|f - f_n\|_{\mathbb{B}} \, \mathrm{d}\mu$ is satisfied, where $\|\ell\|$ stands for the operator norm of ℓ. In particular $\int_E \ell \circ f_n \, \mathrm{d}\mu \to \int_E \ell \circ f \, \mathrm{d}\mu$. Therefore

$$\ell\left(\int_E f \, \mathrm{d}\mu\right) = \lim_{n\to\infty} \ell\left(\int_E f_n \, \mathrm{d}\mu\right) = \lim_{n\to\infty} \int_E \ell \circ f_n \, \mathrm{d}\mu = \int_E \ell \circ f \, \mathrm{d}\mu,$$

proving (6.3) as required. □

Definition 6.3 (Closed operator) *A* closed operator $T : \mathbb{B} \to \mathbb{V}$ *is a couple* $\big(D(T), T\big)$*, where $D(T)$ is a linear subspace of $\mathbb{B}$ and $T : D(T) \to \mathbb{V}$ is a linear map whose graph, defined as* $\mathrm{Graph}(T) := \big\{(v, Tv) \, : \, v \in D(T)\big\}$*, is a closed subspace of the product space $\mathbb{B} \times \mathbb{V}$.*

Closedness of Graph(T) can be equivalently stated as follows: if a sequence $(v_n)_n \subseteq D(T)$ satisfy $\lim_n \|v_n - v\|_{\mathbb{B}} = 0$ and $\lim_n \|Tv_n - w\|_{\mathbb{V}} = 0$ for some vectors $v \in B$ and $w \in \mathbb{V}$, then necessarily $v \in D(T)$ and $w = Tv$.

Example 6.4 (of closed operators) We provide three examples of closed operators:

i) Let $\mathbb{B} = \mathbb{V} = C([0,1])$. Then take $D(T_1) = C^1([0,1])$ and $T_1(f) = f'$.

ii) Let $\mathbb{B} = \mathbb{V} = L^2(0,1)$. Then take $D(T_2) = W^{1,2}(0,1)$ and $T_2(f) = f'$.

iii) Let $\mathbb{B} = L^2(\mathbb{R}^n)$ and $\mathbb{V} = \big[L^2(\mathbb{R}^n)\big]^n$. Then take $D(T_3) = W^{1,2}(\mathbb{R}^n)$ and $T_3(f)$ equal to the n-tuple $(\partial_{x_1} f, \dots, \partial_{x_n} f)$.

Example 6.5 (of non-closed operator) Consider $\mathbb{B} = \mathbb{V} = L^2(\mathbb{R}^n)$, with $n > 1$. Let us define $D(T_4) = W^{1,2}(\mathbb{R}^n)$ and $T_4(f) = \partial_{x_1} f$. Then $\big(D(T_4), T_4\big)$ is not a closed operator.

Exercise 6.6 Prove Example 6.4 and Example 6.5.

Remark 6.7 Let $f \in L^1(\mu; \mathbb{B})$ be given. Suppose that there exists a closed subspace V of $\mathbb{B}$ such that $f(x) \in V$ holds for μ-a.e. $x \in \mathrm{X}$. Then $\int_E f \,\mathrm{d}\mu \in V$ for every $E \subseteq \mathrm{X}$ Borel.

We argue by contradiction: suppose $\int_E f \,\mathrm{d}\mu \notin V$, then we can choose $\ell \in \mathbb{B}'$ with $\ell = 0$ on V and $\ell\big(\int_E f \,\mathrm{d}\mu\big) = 1$ by Hahn-Banach theorem. But the fact that $(\ell \circ f)(x) = 0$ holds for μ-a.e. $x \in \mathrm{X}$ implies that $\ell\big(\int_E f \,\mathrm{d}\mu\big) = \int_E \ell \circ f \,\mathrm{d}\mu = 0$ by (6.3), which is absurd. ■

Theorem 6.8 (Hille) *Let $T : \mathbb{B} \to \mathbb{V}$ be a closed operator. Consider a map $f \in L^1(\mu; \mathbb{B})$ that satisfies $f(x) \in D(T)$ for μ-a.e. $x \in \mathrm{X}$ and $T \circ f \in L^1(\mu; \mathbb{V})$. Then for every $E \subseteq \mathrm{X}$ Borel it holds that $\int_E f \,\mathrm{d}\mu \in D(T)$ and that*

$$T\left(\int_E f \,\mathrm{d}\mu\right) = \int_E T \circ f \,\mathrm{d}\mu. \tag{6.4}$$

Proof. Define the map $\Phi : \mathrm{X} \to \mathbb{B} \times \mathbb{V}$ as $\Phi(x) := \big(f(x), (T \circ f)(x)\big)$ for μ-a.e. $x \in \mathrm{X}$. One can readily check that $\Phi \in L^1(\mu; \mathbb{B} \times \mathbb{V})$. Moreover, $\Phi(x) \in \mathrm{Graph}(T)$ for μ-a.e. $x \in \mathrm{X}$, whence

$$\left(\int_E f \,\mathrm{d}\mu, \int_E T \circ f \,\mathrm{d}\mu\right) = \int_E \Phi(x) \,\mathrm{d}\mu(x) \in \mathrm{Graph}(T)$$

by Remark 6.7. This means that $\int_E f \,\mathrm{d}\mu \in D(T)$ and that $T\big(\int_E f \,\mathrm{d}\mu\big) = \int_E T \circ f \,\mathrm{d}\mu$. □

Let us now concentrate our attention on the case in which $\mathrm{X} = [0,1]$ and $\mu = \mathcal{L}^1|_{[0,1]}$.

Proposition 6.9 *Let $v : [0,1] \to \mathbb{B}$ be an absolutely continuous curve. Suppose that*

$$v'_t := \lim_{h \to 0} \frac{v_{t+h} - v_t}{h} \in \mathbb{B} \qquad \textit{exists for a.e. } t \in [0,1]. \tag{6.5}$$

Then the map $v' : [0,1] \to \mathbb{B}$ is Bochner integrable and satisfies

$$v_t - v_s = \int_s^t v'_r \,\mathrm{d}r \qquad \textit{for every } t, s \in [0,1] \textit{ with } s < t. \tag{6.6}$$

Proof. First of all, by arguing as in Remark 5.1, we see that v' is Borel. Moreover, if V is a closed separable subspace of $\mathbb{B}$ such that $v_t \in V$ for a.e. $t \in [0,1]$, then $v'_t \in V$ for a.e. $t \in [0,1]$ as well, i.e. v' is essentially separably valued. Hence v' is a strongly Borel map. Since the function $\|v'\|_{\mathbb{B}}$ coincides a.e. with the metric speed $|\dot{v}|$, which belongs to $L^1(0,1)$, we conclude that v' is Bochner integrable by Proposition 5.13. Finally, to prove (6.6) it is enough to show that $v_t = v_0 + \int_0^t v'_s \, \mathrm{d}s$ for any $t \in [0,1]$. For every $\ell \in \mathbb{B}'$ it holds that $t \mapsto \ell(v_t) \in \mathbb{R}$ is absolutely continuous, with $\frac{\mathrm{d}}{\mathrm{d}t}\ell(v_t) = \ell(v'_t)$ for a.e. $t \in [0,1]$. Therefore

$$\ell(v_t) = \ell(v_0) + \int_0^t \left(\frac{\mathrm{d}}{\mathrm{d}s}\ell(v_s)\right) \mathrm{d}s = \ell(v_0) + \int_0^t \ell(v'_s) \, \mathrm{d}s \overset{(6.3)}{=} \ell\left(v_0 + \int_0^t v'_s \, \mathrm{d}s\right),$$

which implies that $v_t = v_0 + \int_0^t v'_s \, \mathrm{d}s$ by arbitrariness of $\ell \in \mathbb{B}'$. Thus (6.6) is proved. □

Example 6.10 Let us define the map $v : [0,1] \to L^1(0,1)$ as $v_t := \chi_{[0,t]}$ for every $t \in [0,1]$. Then v is 1-Lipschitz (so also absolutely continuous), because $\|v_t - v_s\|_{L^1(0,1)} = t - s$ holds for every $t, s \in [0,1]$ with $s < t$, but v is not differentiable at any $t \in [0,1]$: the incremental ratios $h^{-1}(v_{t+h} - v_t) = h^{-1}\chi_{(t,t+h]}$ pointwise converge to 0 as $h \searrow 0$ and have $L^1(0,1)$-norm equal to 1. Actually, the probability measures $h^{-1}\chi_{(t,t+h]}\,\mathcal{L}^1$ weakly converges to δ_t as $h \searrow 0$.

Exercise 6.11 Let $\mathbb{B}$ be a Hilbert space (or, more generally, a reflexive Banach space). Prove that any absolutely continuous curve $v : [0,1] \to \mathbb{B}$ is almost everywhere differentiable.

Proposition 6.12 (Lebesgue points) *Let $v : [0,1] \to \mathbb{B}$ be Bochner integrable. Then*

$$\lim_{h \searrow 0} \fint_{t-h}^{t+h} \|v_s - v_t\|_{\mathbb{B}} \, \mathrm{d}s = 0 \qquad \textit{for a.e. } t \in [0,1]. \tag{6.7}$$

Proof. Choose a separable set $V \subseteq \mathbb{B}$ such that $v_t \in V$ for a.e. $t \in [0,1]$ and a sequence $(w_n)_n$ that is dense in V. For any $n \in \mathbb{N}$, the map $t \mapsto \|v_t - w_n\|_{\mathbb{B}} \in \mathbb{R}$ belongs to $L^1(0,1)$, hence there exists a Borel set $N_n \subseteq [0,1]$, with $\mathcal{L}^1(N_n) = 0$, such that

$$\|v_t - w_n\|_{\mathbb{B}} = \lim_{h \searrow 0} \fint_{t-h}^{t+h} \|v_s - w_n\|_{\mathbb{B}} \, \mathrm{d}s \qquad \text{holds for every } t \in [0,1] \setminus N_n,$$

by Lebesgue differentiation theorem. Call $N := \bigcup_n N_n$, which is an $\mathcal{L}^1$-negligible Borel subset of $[0,1]$. Therefore for almost every $t \in [0,1] \setminus N$ one has that

$$\begin{aligned} \varlimsup_{h \searrow 0} \fint_{t-h}^{t+h} \|v_s - v_t\|_{\mathbb{B}} \, \mathrm{d}s &\leq \inf_{n \in \mathbb{N}} \varlimsup_{h \searrow 0} \left[\fint_{t-h}^{t+h} \|v_s - w_n\|_{\mathbb{B}} \, \mathrm{d}s + \|v_t - w_n\|_{\mathbb{B}}\right] \\ &= \inf_{n \in \mathbb{N}} 2\,\|v_t - w_n\|_{\mathbb{B}} = 0 \end{aligned}$$

by density of $(w_n)_n$ in V. Hence (6.7) is proved, getting the thesis. □

7 Lesson [30/10/2017]

Fix two complete and separable metric spaces $(\mathrm{X}, \mathsf{d}_{\mathrm{X}})$, $(\mathrm{Y}, \mathsf{d}_{\mathrm{Y}})$. Let μ and ν be finite Borel measures on X and Y, respectively. In the following three results we will denote by $f : \mathrm{Y} \to \mathbb{R}$ the ν-measurable maps and by $[f]$ the elements of $L^1(\nu)$.

Proposition 7.1 *Let $\mathrm{X} \ni x \mapsto [f_x] \in L^1(\nu)$ be any μ-measurable map. Then there exists a choice $(x, y) \mapsto \tilde{f}(x, y)$ of representatives, i.e. $\big[\tilde{f}(x, \cdot)\big] = [f_x]$ holds for μ-a.e. $x \in \mathrm{X}$, which is Borel measurable. Moreover, any two such choices agree $(\mu \times \nu)$-a.e. in $\mathrm{X} \times \mathrm{Y}$.*

Proof. The thesis is clearly verified whenever $x \mapsto [f_x]$ is a simple map. For $x \mapsto [f_x]$ generic, define $[f^k_x] := \chi_{A_k}(x)\,[f_x]$ for μ-a.e. $x \in \mathrm{X}$, where we put $A_k := \big\{x \in \mathrm{X} \,:\, \big\|[f_x]\big\|_{L^1(\nu)} \leq k\big\}$. Now let $k \in \mathbb{N}$ be fixed. Given that $[f^k]$ belongs to $L^1\big(\mu; L^1(\nu)\big)$, we can choose a sequence of simple maps $[g^n] : \mathrm{X} \to L^1(\nu)$ such that $\big\|[g^n] - [f^k]\big\|_{L^1(\mu;L^1(\nu))} \leq 2^{-2n}$ for every $n \in \mathbb{N}$. As observed in the first part of the proof, we can choose a Borel representative $\tilde{g}^n : \mathrm{X} \times \mathrm{Y} \to \mathbb{R}$ of $[g^n]$ for every $n \in \mathbb{N}$. By using Čebyšëv's inequality, we obtain that

$$\mu\Big(\big\{x \in \mathrm{X} \,:\, \big\|[g^n_x] - [f^k_x]\big\|_{L^1(\nu)} > 2^{-n}\big\}\Big) \leq \frac{1}{2^n} \quad \text{holds for every } n \in \mathbb{N}.$$

Therefore we have that

$$\mu\bigg(\bigcup\nolimits_{n_0 \in \mathbb{N}} \Big\{x \in \mathrm{X} \,:\, \big\|[g^n_x] - [f^k_x]\big\|_{L^1(\nu)} \leq 2^{-n} \text{ for all } n \geq n_0\Big\}\bigg) = \mu(\mathrm{X}). \tag{7.1}$$

Then the functions $\tilde{g}^n$ converge $(\mu \times \nu)$-a.e. to some limit function $\tilde{f}^k : \mathrm{X} \times \mathrm{Y} \to \mathbb{R}$, which is accordingly a Borel representative of $[f^k]$. To conclude, let us define

$$\tilde{f}(x, y) := \sum_{k \in \mathbb{N}} \chi_{A_k \setminus \bigcup_{i<k} A_i}(x)\, \tilde{f}^k(x, y) \quad \text{for every } (x, y) \in \mathrm{X} \times \mathrm{Y}.$$

Therefore $\tilde{f}$ is the desired representative of $x \mapsto [f_x]$, whence the thesis is proved. □

Proposition 7.2 *Consider the operator $\Phi : L^1\big(\mu; L^1(\nu)\big) \to L^1(\mu \times \nu)$ sending $x \mapsto [f_x]$ to (the equivalence class of) one of its Borel representatives $\tilde{f}$ found in Proposition 7.1. Then the map Φ is (well-defined and) an isometric isomorphism.*

Proof. Well-posedness of Φ follows from Proposition 7.1 and from the fact that

$$\big\|[f_\cdot]\big\|_{L^1(\mu;L^1(\nu))} = \iint \big|[f_x]\big|(y)\,\mathrm{d}\nu(y)\,\mathrm{d}\mu(x) = \iint |\tilde{f}|(x, y)\,\mathrm{d}\nu(y)\,\mu(x) = \int |\tilde{f}|\,\mathrm{d}(\mu \times \nu)$$

where the last equality is a consequence of Fubini theorem. The same equalities also guarantee that Φ is an isometry. Moreover, the map Φ is linear, continuous and injective. In order to conclude, it suffices to show that the image of Φ is dense. Given any $\tilde{f} \in C_b(\mathrm{X} \times \mathrm{Y})$, we have that $\lim_{x' \to x} \int \big|\tilde{f}(x', y) - \tilde{f}(x, y)\big|\,\mathrm{d}\nu(y) = 0$ for every $x \in \mathrm{X}$ by dominated convergence theorem, so that $x \mapsto \tilde{f}(x, \cdot) \in L^1(\nu)$ is continuous and accordingly in $L^1\big(\mu; L^1(\nu)\big)$. In other words, we proved that any $\tilde{f} \in C_b(\mathrm{X} \times \mathrm{Y})$ belongs to the image of Φ. Since $C_b(\mathrm{X} \times \mathrm{Y})$ is dense in $L^1(\mu \times \nu)$ by Proposition 2.5, we thus obtained the thesis. □

Proposition 7.3 *Let* $\big(x \mapsto [f_x]\big) \in L^1\big(\mu; L^1(\nu)\big)$ *and call* $[\tilde{f}\,]$ *its image under* Φ*. Then*

$$\left(\int [f_x]\,\mathrm{d}\mu(x)\right)(y) = \int \tilde{f}(x,y)\,\mathrm{d}\mu(x) \quad \text{ holds for } \nu\text{-a.e. } y \in \mathrm{Y}. \tag{7.2}$$

Proof. First of all, we define the linear and continuous operator $T_1 : L^1\big(\mu; L^1(\nu)\big) \to L^1(\nu)$ as $T_1(f) := \int [f_x]\,\mathrm{d}\mu(x) \in L^1(\nu)$ for every $f \in L^1\big(\mu; L^1(\nu)\big)$. On the other hand, by Fubini theorem it makes sense to define $T_2(\tilde{f}) := \big(y \mapsto \int \tilde{f}(x,y)\,\mathrm{d}\mu(x)\big) \in L^1(\nu)$ for all $\tilde{f} \in L^1(\mu \times \nu)$, so that $T_2 : L^1(\mu \times \nu) \to L^1(\nu)$ is a linear and continuous operator. Therefore the diagram

$$\begin{tikzcd} L^1\big(\mu; L^1(\nu)\big) \arrow[r, "\Phi"] \arrow[dr, "T_1"'] & L^1(\mu \times \nu) \arrow[d, "T_2"] \\ & L^1(\nu) \end{tikzcd}$$

is commutative, because T_1 and $T_2 \circ \Phi$ clearly agree on simple maps $f : \mathrm{X} \to L^1(\nu)$. Hence formula (7.2) is proved, as required. $\square$

Lemma 7.4 (Easy version of Dunford-Pettis) *Let* $(f_n)_n \subseteq L^1(\nu)$ *be a sequence with the following property: there exists* $g \in L^1(\nu)$ *such that* $|f_n| \leq g$ *holds* ν*-a.e. for every* $n \in \mathbb{N}$*. Then there exists a subsequence* $(n_k)_k$ *and some function* $f \in L^1(\nu)$ *such that* $f_{n_k} \rightharpoonup f$ *weakly in* $L^1(\nu)$ *and* $|f| \leq g$ *holds* ν*-a.e. in* Y*.*

Proof. For any $k \in \mathbb{N}$, denote $f_n^k := \min\big\{\max\{f_n, -k\}, k\big\}$ and $g_k := \min\big\{\max\{g, -k\}, k\big\}$. The sequence $(f_n^k)_n$ is bounded in $L^2(\nu)$ for any fixed $k \in \mathbb{N}$, thus a diagonalisation argument shows the existence of $(n_i)_i$ and $(h_k)_k \subseteq L^2(\nu)$ such that $f_{n_i}^k \rightharpoonup h_k$ weakly in $L^2(\nu)$ for all k. In particular, $f_{n_i}^k \rightharpoonup h_k$ weakly in $L^1(\nu)$ for all k. Moreover, one can readily check that

$$|f_{n_i}^k - f_{n_i}^{k'}| \leq |g_k - g_{k'}| \ \text{ holds } \nu\text{-a.e.} \quad \text{ for every } i, k, k' \in \mathbb{N}. \tag{7.3}$$

By using (7.3), the lower semicontinuity of $\|\cdot\|_{L^1(\nu)}$ with respect to the weak topology and the dominated convergence theorem, we then deduce that

$$\int |h_k - h_{k'}|\,\mathrm{d}\nu \leq \varliminf_{i\to\infty} \int |f_{n_i}^k - f_{n_i}^{k'}|\,\mathrm{d}\nu \leq \int |g_k - g_{k'}|\,\mathrm{d}\nu \xrightarrow{k,k'} 0, \tag{7.4}$$

which grants that the sequence $(h_k)_k \subseteq L^1(\nu)$ is Cauchy. Call $f \in L^1(\nu)$ its limit. To prove that $f_{n_i} \rightharpoonup f$ weakly in $L^1(\nu)$ as $i \to \infty$, observe that for any $\ell \in L^\infty(\nu)$ it holds that

$$\begin{aligned} \varlimsup_{i\to\infty} \left| \int (f_{n_i} - f)\,\ell\,\mathrm{d}\nu \right| &\leq \varlimsup_{i\to\infty} \left[\int |f_{n_i} - f_{n_i}^k|\,|\ell|\,\mathrm{d}\nu + \left| \int (f_{n_i}^k - h_k)\,\ell\,\mathrm{d}\nu \right| + \int |h_k - f|\,|\ell|\,\mathrm{d}\nu \right] \\ &\leq \Big(\|g - g_k\|_{L^1(\nu)} + \|h_k - f\|_{L^1(\nu)} \Big) \|\ell\|_{L^\infty(\nu)} \\ &\leq 2\,\|g - g_k\|_{L^1(\nu)}\,\|\ell\|_{L^\infty(\nu)} \xrightarrow{k} 0, \end{aligned}$$

where the second inequality stems from (7.3) and the third one from (7.4).

Finally, in order to prove the ν-a.e. inequality $|f| \le g$ it is clearly sufficient to show that

$$\left| \int f \, \ell \, \mathrm{d}\nu \right| \le \int g \, \ell \, \mathrm{d}\nu \quad \text{for every } \ell \in L^\infty(\nu) \text{ with } \ell \ge 0. \tag{7.5}$$

Property (7.5) can be proved by noticing that for any non-negative $\ell \in L^\infty(\nu)$ one has

$$\left| \int f \, \ell \, \mathrm{d}\nu \right| = \lim_{i \to \infty} \left| \int f_{n_i} \, \ell \, \mathrm{d}\nu \right| \le \varliminf_{i \to \infty} \int |f_{n_i}| \, \ell \, \mathrm{d}\nu \le \int g \, \ell \, \mathrm{d}\nu.$$

Therefore the thesis is achieved. □

Hereafter, we shall make use of the following shorthand notation:

$$\mathcal{L}_1 := \mathcal{L}^1|_{[0,1]} \quad \text{and} \quad \Delta := \big\{ (t,s) \in [0,1]^2 \,:\, s \le t \big\}. \tag{7.6}$$

Proposition 7.5 *Let $f : [0,1] \to L^1(\nu)$ and $g \in L^1\big(\mathcal{L}_1; L^1(\nu)\big)$ be given. Suppose that*

$$\big| f_t(y) - f_s(y) \big| \le \int_s^t g_r(y) \, \mathrm{d}r \;\; \textit{holds for } \nu\textit{-a.e. } y \in \mathrm{Y}, \quad \textit{for every } (t,s) \in \Delta. \tag{7.7}$$

Then f is absolutely continuous and $\mathcal{L}_1$-a.e. differentiable. Moreover, its derivative satisfies

$$|f'_t|(y) \le g_t(y) \quad \textit{for } (\mathcal{L}_1 \times \nu)\textit{-a.e. } (t,y) \in [0,1] \times \mathrm{Y}. \tag{7.8}$$

Proof. By integrating (7.7), we get that $\|f_t - f_s\|_{L^1(\nu)} \le \int_s^t \|g_r\|_{L^1(\nu)} \, \mathrm{d}r$ for every $(t,s) \in \Delta$. This proves that $t \mapsto f_t \in L^1(\nu)$ is AC, but in general this does not grant that $t \mapsto f_t$ is a.e. differentiable, cf. for instance Example 6.10. We thus proceed in the following way: let us define $g_t^\varepsilon := \frac{1}{\varepsilon} \int_t^{t+\varepsilon} g_r \, \mathrm{d}r$ for every $\varepsilon > 0$ and $t \in [0,1]$. Observe that

$$\begin{aligned} \|g_\cdot^\varepsilon\|_{L^1(\mathcal{L}_1 \times \nu)} &= \int_0^1 \!\! \int |g_t^\varepsilon|(y) \, \mathrm{d}\nu(y) \, \mathrm{d}t \le \int_0^1 \!\! \int \fint_t^{t+\varepsilon} |g_r|(y) \, \mathrm{d}r \, \mathrm{d}\nu(y) \, \mathrm{d}t \\ &\le \int_0^1 \!\! \int |g_r|(y) \, \mathrm{d}\nu(y) \, \mathrm{d}r = \|g_\cdot\|_{L^1(\mathcal{L}_1 \times \nu)} \end{aligned} \tag{7.9}$$

is satisfied for every $\varepsilon > 0$. Given any map $h \in C\big([0,1], L^1(\nu)\big)$, it clearly holds that $h_\cdot^\varepsilon \to h_\cdot$ in $L^1(\mathcal{L}_1 \times \nu)$ as $\varepsilon \searrow 0$. Therefore for any such h one has that

$$\begin{aligned} \varlimsup_{\varepsilon \searrow 0} \|g^\varepsilon - g\|_{L^1(\mathcal{L}_1 \times \nu)} &\le \varlimsup_{\varepsilon \searrow 0} \Big[\big\|(g-h)^\varepsilon\big\|_{L^1(\mathcal{L}_1 \times \nu)} + \|h^\varepsilon - h\|_{L^1(\mathcal{L}_1 \times \nu)} \Big] + \|h - g\|_{L^1(\mathcal{L}_1 \times \nu)} \\ &\le 2 \, \|g - h\|_{L^1(\mathcal{L}_1 \times \nu)} + \varlimsup_{\varepsilon \searrow 0} \|h^\varepsilon - h\|_{L^1(\mathcal{L}_1 \times \nu)} \\ &= 2 \, \|g - h\|_{L^1(\mathcal{L}_1 \times \nu)}, \end{aligned}$$

where the second inequality follows from (7.9) and the third one from continuity of h. Given that $C\big([0,1], L^1(\nu)\big)$ is dense in $L^1\big(\mathcal{L}_1; L^1(\nu)\big)$, we conclude that $\lim_{\varepsilon \searrow 0} \|g^\varepsilon - g\|_{L^1(\mathcal{L}_1 \times \nu)} = 0$.

In particular, there exist a sequence $\varepsilon_n \searrow 0$ and a function $G \in L^1(\mathcal{L}_1 \times \nu)$ such that the inequality $g^{\varepsilon_n} \le G$ holds $(\mathcal{L}_1 \times \nu)$-a.e. for every $n \in \mathbb{N}$. This grant that

$$\left| \frac{f_{t+\varepsilon_n} - f_t}{\varepsilon_n} \right| \le \frac{1}{\varepsilon_n} \int_t^{t+\varepsilon_n} g_r \, \mathrm{d}r = g_t^{\varepsilon_n} \le G_t \;\; \text{holds } \nu\text{-a.e.} \quad \text{for a.e. } t \in [0,1]. \tag{7.10}$$

The bound in (7.10) allows us to apply Lemma 7.4: up to a not relabeled subsequence, we have that $(f_{\cdot+\varepsilon_n} - f_\cdot)/\varepsilon_n$ weakly converges in $L^1(\mathcal{L}_1 \times \nu)$ to some function $f' \in L^1(\mathcal{L}_1 \times \nu)$. Moreover, simple computations yield

$$\int_s^t \frac{f_{r+\varepsilon_n} - f_r}{\varepsilon_n}\,\mathrm{d}r = ⨍_t^{t+\varepsilon_n} f_r\,\mathrm{d}r - ⨍_s^{s+\varepsilon_n} f_r\,\mathrm{d}r \quad \text{for every } (t,s) \in \Delta. \tag{7.11}$$

The continuity of $r \mapsto f_r \in L^1(\nu)$ grants that the right hand side in (7.11) converges to $f_t - f_s$ in $L^1(\nu)$ as $n \to \infty$. On the other hand, for every $\ell \in L^\infty(\nu)$ it holds that

$$\int \ell(y)\left(\int_s^t \frac{f_{r+\varepsilon_n} - f_r}{\varepsilon_n}\,\mathrm{d}r\right)(y)\,\mathrm{d}\nu(y) = \int \underbrace{\ell(y)\,\chi_{[s,t]}(r)}_{\in L^\infty(\mathcal{L}_1\times\nu)} \frac{f_{r+\varepsilon_n}(y) - f_r(y)}{\varepsilon_n}\,\mathrm{d}(\mathcal{L}_1 \times \nu)(r,y),$$

which in turn converges to $\int \ell(y)\left(\int_s^t f'_r\,\mathrm{d}r\right)(y)\,\mathrm{d}\nu(y)$ as $n \to \infty$. In other words, we showed that $\int_s^t (f_{r+\varepsilon_n} - f_r)/\varepsilon_n\,\mathrm{d}r \rightharpoonup \int_s^t f'_r\,\mathrm{d}r$ weakly in $L^1(\nu)$. So by letting $n \to \infty$ in (7.11) we get

$$\int_s^t f'_r\,\mathrm{d}r = f_t - f_s \quad \text{for every } (t,s) \in \Delta.$$

Therefore Proposition 6.12 implies that f'_t is the strong derivative in $L^1(\nu)$ of the map $t \mapsto f_t$ for a.e. $t \in [0,1]$. Finally, by recalling (7.7) we also conclude that (7.8) is verified. □

Lemma 7.6 *Let $h \in L^1(0,1)$ be given. Then $h \in W^{1,1}(0,1)$ if and only if there exists a function $g \in L^1(0,1)$ such that*

$$h_t - h_s = \int_s^t g_r\,\mathrm{d}r \quad \textit{holds for } \mathcal{L}^2\textit{-a.e. } (t,s) \in \Delta. \tag{7.12}$$

Moreover, in such case it holds that $h' = g$.

Proof. Necessity. Fix any family of convolution kernels $\rho_\varepsilon \in C_c^\infty(\mathbb{R})$, i.e. $\int \rho_\varepsilon(x)\,\mathrm{d}x = 1$, the support of ρ_ε is contained in $(-\varepsilon,\varepsilon)$ and $\rho_\varepsilon \geq 0$. Let us define $h^\varepsilon := h * \rho_\varepsilon$ for all $\varepsilon > 0$. Recall that $h^\varepsilon \in C_c^\infty(\mathbb{R})$ and that $(h^\varepsilon)' = (h') * \rho_\varepsilon$. Choose a sequence $\varepsilon_n \searrow 0$ and a negligible Borel set $N \subseteq [0,1]$ such that $h_t^{\varepsilon_n} \to h_t$ as $n \to \infty$ for every $t \in [0,1] \setminus N$. Given that we have the equality $h_t^{\varepsilon_n} - h_s^{\varepsilon_n} = \int_s^t (h^{\varepsilon_n})'_r\,\mathrm{d}r$ for every $n \in \mathbb{N}$ and $(t,s) \in \Delta$, we can finally conclude that $h_t - h_s = \int_s^t h'_r\,\mathrm{d}r$ for $\mathcal{L}^2$-a.e. $(t,s) \in \Delta$, proving (7.12) with $g = h'$.
Sufficiency. By Fubini theorem, we see that for a.e. $\varepsilon > 0$ it holds that $h_{t+\varepsilon} - h_t = \int_t^{t+\varepsilon} g_r\,\mathrm{d}r$ for a.e. $t \in [0,1]$. In particular, there is a sequence $\varepsilon_n \searrow 0$ such that $h_{t+\varepsilon_n} - h_t = \int_t^{t+\varepsilon_n} g_r\,\mathrm{d}r$ for every $n \in \mathbb{N}$ and for a.e. $t \in [0,1]$. Now fix $\varphi \in C_c^\infty(0,1)$. Then

$$\int \frac{\varphi_{t-\varepsilon_n} - \varphi_t}{\varepsilon_n}\,h_t\,\mathrm{d}t = \int \frac{h_{t+\varepsilon_n} - h_t}{\varepsilon_n}\,\varphi_t\,\mathrm{d}t = \int\left(⨍_t^{t+\varepsilon_n} g_r\,\mathrm{d}r\right)\varphi_t\,\mathrm{d}t. \tag{7.13}$$

By applying the dominated convergence theorem, we finally deduce by letting $n \to \infty$ in the equation (7.13) that $-\int \varphi'_t\,h_t\,\mathrm{d}t = \int g_t\,\varphi_t\,\mathrm{d}t$. Hence $h \in W^{1,1}(0,1)$ and $h' = g$. □

We are now in a position to prove Theorem 4.18. For the sake of clarity, we restate it:

Theorem 7.7 (Theorem 4.18) *Consider a metric measure space* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *as in* (1.1)*. Fix a Borel map* $f:\mathrm{X}\to\mathbb{R}$*. Let* $G\in L^2(\mathfrak{m})$ *satisfy* $G\geq 0$*. Then the following are equivalent:*

i) $f\in \mathrm{S}^2(\mathrm{X})$ *and* G *is a weak upper gradient of* f*.*

ii) *For any test plan* $\boldsymbol{\pi}$*, we have that* $t\mapsto f\circ \mathrm{e}_t - f\circ \mathrm{e}_0\in L^1(\boldsymbol{\pi})$ *is AC. For a.e.* $t\in[0,1]$*, there exists the strong* $L^1(\boldsymbol{\pi})$*-limit of* $(f\circ\mathrm{e}_{t+h}-f\circ\mathrm{e}_t)/h$ *as* $h\to 0$*. Such limit, denoted by* $\mathsf{Der}_{\boldsymbol{\pi}}(f)_t\in L^1(\boldsymbol{\pi})$*, satisfies* $\big|\mathsf{Der}_{\boldsymbol{\pi}}(f)_t\big|(\gamma)\leq G(\gamma_t)|\dot\gamma_t|$ *for* $(\boldsymbol{\pi}\times\mathcal{L}_1)$*-a.e.* (γ,t)*.*

iii) *For every test plan* $\boldsymbol{\pi}$*, we have for* $\boldsymbol{\pi}$*-a.e.* γ *that* $f\circ\gamma$ *belongs to* $W^{1,1}(0,1)$ *and that the inequality* $\big|(f\circ\gamma)'_t\big|\leq G(\gamma_t)|\dot\gamma_t|$ *holds for a.e.* $t\in[0,1]$*.*

If the above hold, then the equality $\mathsf{Der}_{\boldsymbol{\pi}}(f)_t(\gamma)=(f\circ\gamma)'_t$ *is verified for* $(\boldsymbol{\pi}\times\mathcal{L}_1)$*-a.e.* (γ,t)*.*

Proof.
i) $\Longrightarrow$ ii) We have that $\big|f(\gamma_t)-f(\gamma_s)\big|\leq\int_s^t G(\gamma_r)|\dot\gamma_r|\,\mathrm{d}r$ is satisfied for every $(t,s)\in\Delta$ and for $\boldsymbol{\pi}$-a.e. γ by Proposition 5.7. Since the map $(\gamma,t)\mapsto G(\gamma_t)|\dot\gamma_t|$ belongs to $L^1(\boldsymbol{\pi}\times\mathcal{L}_1)$ by Remark 4.8 and Remark 5.1, we obtain ii) by applying Proposition 7.5.
ii) $\Longrightarrow$ iii) By Fubini theorem, one has for $\boldsymbol{\pi}$-a.e. γ that $f(\gamma_t)-f(\gamma_s)=\int_s^t \mathrm{Der}_{\boldsymbol{\pi}}(f)_r(\gamma)\,\mathrm{d}r$ holds for $\mathcal{L}^2$-a.e. $(t,s)\in\Delta$, whence iii) stems from Lemma 7.6. Further, for $\boldsymbol{\pi}$-a.e. γ we have

$$\int_s^t (f\circ\gamma)'_r\,\mathrm{d}r=f(\gamma_t)-f(\gamma_s)=\int_s^t \mathrm{Der}_{\boldsymbol{\pi}}(f)_r(\gamma)\,\mathrm{d}r\qquad\text{for }\mathcal{L}^2\text{-a.e. }(t,s)\in\Delta,$$

which in turn implies the last statement of the theorem.
iii) $\Longrightarrow$ i) Fix a test plan $\boldsymbol{\pi}$ on X. Choose a point $\bar x\in\mathrm{X}$ and a sequence of 1-Lipschitz functions $(\eta_n)_n\subseteq C_b(\mathrm{X})$ such that $\eta_n=1$ on $B_n(\bar x)$ and $\mathrm{spt}(\eta_n)\subseteq B_{n+2}(\bar x)$. Let us define

$$f^{mn}:=\eta_n\,\min\big\{\max\{f,-m\},m\big\}\qquad\text{for every }m,n\in\mathbb{N}.$$

Fix $m,n\in\mathbb{N}$. Notice that $f^{mn}\circ\gamma\in W^{1,1}(0,1)$ for $\boldsymbol{\pi}$-a.e. γ, so that Lemma 7.6 implies that

$$\int\big|f^{mn}(\gamma_t)-f^{mn}(\gamma_s)\big|\,\mathrm{d}\boldsymbol{\pi}(\gamma)\leq\iint_s^t\big|(f^{mn}\circ\gamma)'_r\big|\,\mathrm{d}r\,\mathrm{d}\boldsymbol{\pi}(\gamma)\qquad\text{for }\mathcal{L}^2\text{-a.e. }(t,s)\in\Delta.\tag{7.14}$$

The right hand side in (7.14) is clearly continuous in (t,s). Since $f^{mn}\in L^1(\mathfrak{m})$, we deduce from Proposition 5.4 that also the left hand side is continuous in (t,s), thus in particular

$$\int\big|f^{mn}(\gamma_1)-f^{mn}(\gamma_0)\big|\,\mathrm{d}\boldsymbol{\pi}(\gamma)\leq\iint_0^1\big|(f^{mn}\circ\gamma)'_t\big|\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}(\gamma).\tag{7.15}$$

Moreover, $\big|(f^{mn}\circ\gamma)'_t\big|\leq m\,|\dot\gamma_t|\,\chi_{B_n(\bar x)^c}(\gamma_t)+\big|(f\circ\gamma)'_t\big|$ is satisfied for $(\boldsymbol{\pi}\times\mathcal{L}_1)$-a.e. (γ,t) as a consequence of the Leibniz rule, whence

$$\begin{aligned}\int\big|f(\gamma_1)-f(\gamma_0)\big|\,\mathrm{d}\boldsymbol{\pi}(\gamma)&\leq\varliminf_{m\to\infty}\varliminf_{n\to\infty}\int\big|f^{mn}(\gamma_1)-f^{mn}(\gamma_0)\big|\,\mathrm{d}\boldsymbol{\pi}(\gamma)\\&\leq\varliminf_{m\to\infty}\varliminf_{n\to\infty}\iint_0^1\Big[m\,|\dot\gamma_t|\,\chi_{B_n(\bar x)^c}(\gamma_t)+\big|(f\circ\gamma)'_t\big|\Big]\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}(\gamma)\\&=\varliminf_{m\to\infty}\iint_0^1\big|(f\circ\gamma)'_t\big|\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}(\gamma)\leq\iint_0^1 G(\gamma_t)|\dot\gamma_t|\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}(\gamma),\end{aligned}$$

where the first line follows from Fatou lemma, the second one from (7.15) and the third one from the dominated convergence theorem. Therefore i) is proved. $\square$

Remark 7.8 To be more precise, the last statement in Theorem 7.7 should be stated in the following way: we can choose a Borel representative $F \in L^1(\mathcal{L}_1 \times \boldsymbol{\pi})$ of $t \mapsto \mathsf{Der}_{\boldsymbol{\pi}}(f)_t \in L^1(\boldsymbol{\pi})$ in the sense of Proposition 7.1, since such map belongs to $L^1\big(\mathcal{L}_1; L^1(\boldsymbol{\pi})\big)$ by ii). Analogously, we can choose a Borel representative $\tilde{F} \in L^1(\boldsymbol{\pi} \times \mathcal{L}_1)$ of $\gamma \mapsto \big(t \mapsto (f \circ \gamma)'_t \in L^1(0,1)\big)$, which belongs to $L^1\big(\boldsymbol{\pi}; L^1(\mathcal{L}_1)\big)$ by iii). Then $F(t,\gamma) = \tilde{F}(\gamma,t)$ holds for $(\boldsymbol{\pi} \times \mathcal{L}_1)$-a.e. (γ,t). $\blacksquare$

8 Lesson [06/11/2017]

We point out some consequences of Theorem 7.7, already mentioned in Remark 4.19:

Proposition 8.1 *Let $f \in \mathrm{S}^2(\mathrm{X})$. Consider two weak upper gradients $G_1, G_2 \in L^2(\mathfrak{m})$ of f. Then $\min\{G_1, G_2\}$ is a weak upper gradient of f.*

Proof. By point ii) of Theorem 7.7, we know that $\big|\mathrm{Der}_{\boldsymbol{\pi}}(f)_t\big|(\gamma) \leq G_i(\gamma_t)|\dot{\gamma}_t|$ holds for $i = 1,2$ and for $(\boldsymbol{\pi} \times \mathcal{L}_1)$-a.e. (γ,t), thus also $\big|\mathrm{Der}_{\boldsymbol{\pi}}(f)_t\big|(\gamma) \leq (G_1 \wedge G_2)(\gamma_t)|\dot{\gamma}_t|$ for $(\boldsymbol{\pi} \times \mathcal{L}_1)$-a.e. (γ,t). Therefore $G_1 \wedge G_2$ is a weak upper gradient of f, again by Theorem 7.7. $\square$

Corollary 8.2 *Let $f \in \mathrm{S}^2(\mathrm{X})$. Let $G \in L^2(\mathfrak{m})$ be a weak upper gradient of f. Then $|Df| \leq G$ holds $\mathfrak{m}$-a.e. in X. In other words, $|Df|$ is minimal also in the $\mathfrak{m}$-a.e. sense.*

Proof. We argue by contradiction: suppose that there exists a weak upper gradient G of f such that $\mathfrak{m}\big(\{G < |Df|\}\big) > 0$. Hence the function $G \wedge |Df|$, which has an $L^2(\mathfrak{m})$-norm that is strictly smaller than $\big\||Df|\big\|_{L^2(\mathfrak{m})}$, is a weak upper gradient of f by Proposition 8.1. This leads to a contradiction, thus proving the statement. $\square$

Given any $f \in \mathrm{LIP}(\mathrm{X})$, we define the *local Lipschitz constant* $\mathrm{lip}(f) : \mathrm{X} \to [0,+\infty)$ as

$$\mathrm{lip}(f)(x) := \varlimsup_{y \to x} \frac{\big|f(y) - f(x)\big|}{\mathsf{d}(y,x)} \qquad \text{if } x \in \mathrm{X} \text{ is an accumulation point} \tag{8.1}$$

and $\mathrm{lip}(f)(x) := 0$ otherwise.

Remark 8.3 Given a Lipschitz function $f \in \mathrm{LIP}(\mathrm{X})$ and an AC curve $\gamma : [0,1] \to \mathrm{X}$, it holds that $t \mapsto f(\gamma_t) \in \mathbb{R}$ is AC and satisfies

$$\big|(f \circ \gamma)'_t\big| \leq \mathrm{lip}(f)(\gamma_t)\,|\dot{\gamma}_t| \qquad \text{for a.e. } t \in [0,1]. \tag{8.2}$$

Indeed, to check that $f \circ \gamma$ is AC simply notice that $\big|f(\gamma_t) - f(\gamma_s)\big| \leq \mathrm{Lip}(f) \int_s^t |\dot{\gamma}_r|\,\mathrm{d}r$ holds for any $t,s \in [0,1]$ with $s \leq t$. Now fix $t \in [0,1]$ such that both $(f \circ \gamma)'_t$ and $|\dot{\gamma}_t|$ exist (which holds for a.e. t). If γ is constant in some neighbourhood of t, then (8.2) is trivially verified (since the left hand side is null). In the remaining case, we have that

$$(f \circ \gamma)'_t = \lim_{h \to 0} \frac{\big|(f \circ \gamma)_{t+h} - (f \circ \gamma)_t\big|}{|h|} \leq \varlimsup_{h \to 0} \frac{\big|f(\gamma_{t+h}) - f(\gamma_t)\big|}{\mathsf{d}(\gamma_{t+h},\gamma_t)} \lim_{h \to 0} \frac{\mathsf{d}(\gamma_{t+h},\gamma_t)}{|h|} \leq \mathrm{lip}(f)(\gamma_t)\,|\dot{\gamma}_t|,$$

obtaining (8.2). $\blacksquare$

Proposition 8.4 *Let $f \in \mathrm{LIP}_{bs}(\mathrm{X})$ be given. Then $f \in \mathrm{S}^2(\mathrm{X})$ and $|Df| \leq \mathrm{lip}(f) \leq \mathrm{Lip}(f)$ holds $\mathfrak{m}$-a.e. in* X.

Proof. For any AC curve γ, we have that $\big|f(\gamma_1) - f(\gamma_0)\big| \leq \int_0^1 \mathrm{lip}(f)(\gamma_t)\,|\dot\gamma_t|\,\mathrm{d}t$ by (8.2). By integrating such inequality with respect to any test plan $\boldsymbol{\pi}$, we get the thesis. □

Definition 8.5 (Upper gradient) *Consider two functions $f, g : \mathrm{X} \to \mathbb{R}$, with $g \geq 0$. Then we say that g is an* upper gradient *of f provided for any AC curve $\gamma : [0,1] \to \mathrm{X}$ one has that the curve $f \circ \gamma$ is AC and satisfies $\big|(f\circ\gamma)'_t\big| \leq g(\gamma_t)|\dot\gamma_t|$ for a.e. $t \in [0,1]$.*

Note that $\mathrm{lip}(f)$ is an upper gradient of f for any $f \in \mathrm{LIP}(\mathrm{X})$, as shown in Remark 8.3.

Remark 8.6 In general, a 'minimal upper gradient' might fail to exist. ■

Theorem 8.7 *The following hold:*

A) Locality. *Let $f, g \in \mathrm{S}^2(\mathrm{X})$ be given. Then $|Df| = |Dg|$ holds $\mathfrak{m}$-a.e. in $\{f = g\}$.*

B) Chain rule. *Let $f \in \mathrm{S}^2(\mathrm{X})$ be given.*

   B1) *If a Borel set $N \subseteq \mathbb{R}$ is $\mathcal{L}^1$-negligible, then $|Df| = 0$ holds $\mathfrak{m}$-a.e. in $f^{-1}(N)$.*

   B2) *If $\varphi : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function, then $\varphi\circ f \in \mathrm{S}^2(\mathrm{X})$ and $|D(\varphi\circ f)| = |\varphi'|\circ f\,|Df|$ holds $\mathfrak{m}$-a.e., where $|\varphi'| \circ f$ is arbitrarily defined on $f^{-1}\big(\{t \in \mathbb{R} : \nexists\,\varphi'(t)\}\big)$.*

C) Leibniz rule. *Let $f, g \in \mathrm{S}^2(\mathrm{X}) \cap L^\infty(\mathfrak{m})$ be given. Then $fg \in \mathrm{S}^2(\mathrm{X}) \cap L^\infty(\mathfrak{m})$ and the inequality $|D(fg)| \leq |f||Dg| + |g||Df|$ holds $\mathfrak{m}$-a.e. in* X.

Proof. Step 1. First of all, we claim that

$$f \in \mathrm{S}^2(\mathrm{X}),\ \varphi \in \mathrm{LIP}(\mathbb{R}) \quad\Longrightarrow\quad \varphi\circ f \in \mathrm{S}^2(\mathrm{X}),\ |D(\varphi\circ f)| \leq \mathrm{Lip}(\varphi)|Df| \ \ \mathfrak{m}\text{-a.e.}. \tag{8.3}$$

Indeed, the inequality $\int\big|(\varphi\circ f)(\gamma_1) - (\varphi\circ f)(\gamma_0)\big|\,\mathrm{d}\boldsymbol{\pi}(\gamma) \leq \mathrm{Lip}(\varphi)\iint_0^1 |Df|(\gamma_t)|\dot\gamma_t|\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}(\gamma)$ holds for any test plan $\boldsymbol{\pi}$, thus proving (8.3).
Step 2. Given $h \in W^{1,1}(0,1)$ and $\varphi \in C^1(\mathbb{R}) \cap \mathrm{LIP}(\mathbb{R})$, we have that $\varphi\circ h \in W^{1,1}(0,1)$ and that $(\varphi\circ h)' = \varphi'\circ h\,h'$ holds a.e. in $(0,1)$. In order to prove it, call $h_\varepsilon := h * \rho_\varepsilon$ for all $\varepsilon > 0$, notice that $(\varphi\circ h_\varepsilon)' = \varphi'\circ h_\varepsilon\,h'_\varepsilon$ because h_ε is smooth and finally pass to the limit as $\varepsilon \searrow 0$.
Step 3. We now claim that

$$f \in \mathrm{S}^2(\mathrm{X}),\ \varphi \in C^1(\mathbb{R}) \cap \mathrm{LIP}(\mathbb{R}) \quad\Longrightarrow\quad |D(\varphi\circ f)| \leq |\varphi'|\circ f\,|Df| \ \ \mathfrak{m}\text{-a.e.}. \tag{8.4}$$

To prove it: fix a test plan $\boldsymbol{\pi}$. For $\boldsymbol{\pi}$-a.e. γ, it holds that $t \mapsto f(\gamma_t)$ belongs to $W^{1,1}(0,1)$ and that $\big|(f\circ\gamma)'_t\big| \leq |Df|(\gamma_t)|\dot\gamma_t|$ for a.e. $t \in [0,1]$, by Theorem 7.7. Hence Step 2 grants that the function $t \mapsto (\varphi\circ f)(\gamma_t)$ is in $W^{1,1}(0,1)$ and satisfies

$$\big|(\varphi\circ f\circ\gamma)'_t\big| \leq \big(|\varphi'|\circ f\big)(\gamma_t)\,\big|(f\circ\gamma)'_t\big| \leq \big(|\varphi'|\circ f\big)(\gamma_t)\,|Df|(\gamma_t)\,|\dot\gamma_t| \qquad \text{for a.e. } t \in [0,1],$$

whence $|D(\varphi \circ f)| \leq |\varphi'| \circ f\, |Df|$ holds $\mathfrak{m}$-a.e. by Theorem 7.7, thus proving (8.4).
STEP 4. We want to show that

$$f \in \mathrm{S}^2(X),\ K \subseteq \mathbb{R} \text{ compact with } \mathcal{L}^1(K) = 0 \quad \Longrightarrow \quad |Df| = 0 \ \ \mathfrak{m}\text{-a.e. in } f^{-1}(K). \tag{8.5}$$

For any $n \in \mathbb{N}$, let us call $\psi_n := n\, \mathsf{d}(\cdot, K) \wedge 1$. Since the $\mathcal{L}^1$-measure of the ε-neighbourhood of K converges to 0 as $\varepsilon \searrow 0$, we deduce that $\mathcal{L}^1\big(\{\psi_n < 1\}\big) \to 0$ as $n \to \infty$. Now call φ_n the primitive of ψ_n. Given that ψ_n is continuous and bounded, we have that φ_n is C^1 and Lipschitz. Moreover, it holds that φ_n uniformly converges to $\mathrm{id}_{\mathbb{R}}$ as $n \to \infty$, because

$$\big|\varphi_n(t) - t\big| \leq \int_0^t \big|\psi_n(s) - 1\big| \, \mathrm{d}s \leq \mathcal{L}^1\big(\{\psi_n < 1\}\big) \xrightarrow{n} 0.$$

In particular $\varphi_n \circ f \to f$ pointwise $\mathfrak{m}$-a.e., whence Proposition 5.2 gives

$$\int |Df|^2 \, \mathrm{d}\mathfrak{m} \leq \varliminf_{n \to \infty} \int |D(\varphi_n \circ f)|^2 \, \mathrm{d}\mathfrak{m} \overset{(8.4)}{\leq} \varliminf_{n \to \infty} \int |\varphi_n'|^2 \circ f\, |Df|^2 \, \mathrm{d}\mathfrak{m} \leq \int_{\mathrm{X} \setminus f^{-1}(K)} |Df|^2 \, \mathrm{d}\mathfrak{m},$$

where in the last inequality we used the facts that $|\varphi_n'| \leq \|\psi_n\|_{L^\infty(\mathbb{R})} = 1$ and that $\varphi_n' = \psi_n = 0$ on K. This forces $|Df|$ to be $\mathfrak{m}$-a.e. null in the set $f^{-1}(K)$, obtaining (8.5).
STEP 5. We now use STEP 4 to prove B1). Take $f \in \mathrm{S}^2(\mathrm{X})$ and $N \subseteq \mathbb{R}$ Borel with $\mathcal{L}^1(N) = 0$. There exists a measure $\widetilde{\mathfrak{m}} \in \mathscr{P}(\mathrm{X})$ such that $\mathfrak{m} \ll \widetilde{\mathfrak{m}} \ll \mathfrak{m}$, in other words having exactly the same negligible sets as $\mathfrak{m}$. For instance, choose any Borel partition $(B_n)_{n \geq 1}$ of the space X such that $0 < \mathfrak{m}(B_n) < +\infty$ for every $n \in \mathbb{N}$ and define

$$\widetilde{\mathfrak{m}} := \sum_{n=1}^\infty \frac{1}{2^n\, \mathfrak{m}(B_n)}\, \mathfrak{m}|_{B_n}.$$

Now let us call $\mu := f_* \widetilde{\mathfrak{m}}$. Since $\widetilde{\mathfrak{m}}$ is finite, we have that μ is a Radon measure on $\mathbb{R}$, in particular μ is inner regular. Then there exists a sequence $(K_n)_n$ of compact subsets of N such that $\mu\big(N \setminus \bigcup_n K_n\big) = 0$, or equivalently $\mathfrak{m}\big(f^{-1}\big(N \setminus \bigcup_n K_n\big)\big) = 0$. Given that $|Df| = 0$ is verified $\mathfrak{m}$-a.e. in $\bigcup_n f^{-1}(K_n) = f^{-1}\big(\bigcup_n K_n\big)$ by (8.5), we thus conclude that B1) is satisfied.
STEP 6. We claim that

$$f \in \mathrm{S}^2(\mathrm{X}),\ \varphi \in \mathrm{LIP}(\mathbb{R}) \quad \Longrightarrow \quad |D(\varphi \circ f)| \leq |\varphi'| \circ f\, |Df| \ \ \mathfrak{m}\text{-a.e.}. \tag{8.6}$$

To prove it, call $\varphi_n := \varphi * \rho_{1/n}$. Up to a not relabeled subsequence, we have that $\varphi_n \to \varphi$ pointwise and $\varphi_n' \to \varphi'$ a.e.. Denote by N the (negligible) set of $t \in \mathbb{R}$ such that either φ is not differentiable at t, or $\lim_n \varphi_n'(t)$ does not exist, or $\varphi'(t)$ and $\lim_n \varphi_n'(t)$ exist but are different. We know that $|D(\varphi_n \circ f)| \leq |\varphi_n'| \circ f\, |Df|$ holds $\mathfrak{m}$-a.e. for all $n \in \mathbb{N}$ by (8.4). Given that the inequality $|\varphi_n'| \circ f\, |Df| \leq \mathrm{Lip}(\varphi)|Df|$ is satisfied $\mathfrak{m}$-a.e. for every $n \in \mathbb{N}$, we can thus deduce that $|\varphi_n'| \circ f\, |Df| \to |\varphi'| \circ f\, |Df|$ in $L^2(\mathfrak{m})$ by B1) and dominated convergence theorem. Moreover, one has that $\varphi_n \circ f \to \varphi \circ f$ in the $\mathfrak{m}$-a.e. sense, whence $|D(\varphi \circ f)| \leq |\varphi'| \circ f\, |Df|$ holds $\mathfrak{m}$-a.e. by Proposition 4.11. This proves the claim (8.6).

STEP 7. We now deduce property B2) from (8.6). Suppose wlog that $\mathrm{Lip}(\varphi) = 1$. Let us define $\psi^{\pm}(t) := \pm t - \varphi(t)$ for every $t \in \mathbb{R}$. Then $\mathfrak{m}$-a.e. in the set $f^{-1}\big(\{\pm\varphi' \geq 0\}\big)$ we have

$$|Df| = |D(\pm f)| \leq |D(\varphi \circ f)| + |D(\psi^{\pm} \circ f)| \leq \big(|\varphi'| \circ f + |(\psi^{\pm})'| \circ f\big)\, |Df| = |Df|,$$

which forces $|D(\varphi \circ f)| = \pm\varphi' \circ f\, |Df|$ to hold $\mathfrak{m}$-a.e. in $f^{-1}\big(\{\pm\varphi' \geq 0\}\big)$, which is B2).

STEP 8. Property A) readily follows from B1): if $h := f - g$ then $\big||Df| - |Dg|\big| \leq |Dh| = 0$ holds $\mathfrak{m}$-a.e. in $h^{-1}(\{0\}) = \{f = g\}$ by B1), proving A).

STEP 9. We conclude by deducing C) from B2). Given two functions $h_1, h_2 \in W^{1,1}(0,1)$, we have that $h_1 h_2 \in W^{1,1}(0,1)$ and $(h_1 h_2)' = h_1' h_2 + h_1 h_2'$. Now fix $f, g \in \mathrm{S}^2(\mathrm{X}) \cap L^\infty(\mathfrak{m})$. Given any test plan $\boldsymbol{\pi}$, we have for $\boldsymbol{\pi}$-a.e. γ that $f \circ \gamma, g \circ \gamma \in W^{1,1}(0,1)$, so that $(fg) \circ \gamma \in W^{1,1}(0,1)$ as well. Further, $\big|(f \circ \gamma)_t'\big| \leq |Df|(\gamma_t)|\dot{\gamma}_t|$ and $\big|(g \circ \gamma)_t'\big| \leq |Dg|(\gamma_t)|\dot{\gamma}_t|$ for a.e. $t \in [0,1]$, whence

$$\big|\big((fg) \circ \gamma\big)_t'\big| \leq |f|(\gamma_t)\, \big|(g \circ \gamma)_t'\big| + |g|(\gamma_t)\, \big|(f \circ \gamma)_t'\big| \leq \underbrace{\big[|f||Dg| + |g||Df|\big]}_{\in L^2(\mathfrak{m})}(\gamma_t)\, |\dot{\gamma}_t|$$

is satisfied for a.e. $t \in [0,1]$. Therefore $fg \in \mathrm{S}^2(\mathrm{X})$ and $|f||Dg| + |g||Df|$ is a weak upper gradient of fg by Theorem 7.7, thus proving C). □

Remark 8.8 We present an alternative proof of property C) of Theorem 8.7:
First of all, suppose that $f, g \geq c$ for some constant $c > 0$. Note that the function log is Lipschitz in $[c, +\infty)$, then choose any Lipschitz function $\varphi : \mathbb{R} \to \mathbb{R}$ that coincides with log in $[c, +\infty)$. Now call $C := \log\big(\|fg\|_{L^\infty(\mathfrak{m})}\big)$ and choose a Lipschitz function $\psi : \mathbb{R} \to \mathbb{R}$ such that $\psi = \exp$ in the interval $\big[\log(c^2), C\big]$. By applying property B2) of Theorem 8.7, we see that $\varphi \circ (fg) = \log(fg) = \log(f) + \log(g) = \varphi \circ f + \varphi \circ g$ belongs to $\mathrm{S}^2(\mathrm{X})$ and accordingly that $fg = \exp\big(\log(fg)\big) = \psi \circ \varphi \circ (fg) \in \mathrm{S}^2(\mathrm{X})$. Furthermore, again by B2) we deduce that

$$\begin{aligned} |D(fg)| &= |\psi'| \circ \varphi \circ (fg)\, \big|D\big(\varphi \circ (fg)\big)\big| \leq |fg|\, \Big[\big|D\log(f)\big| + \big|D\log(g)\big|\Big] \\ &= |fg|\, \left[\frac{|Df|}{|f|} + \frac{|Dg|}{|g|}\right] = |f||Dg| + |g||Df| \qquad \mathfrak{m}\text{-a.e. in } \mathrm{X}. \end{aligned}$$

Now consider the case of general $f, g \in \mathrm{S}^2(\mathrm{X}) \cap L^\infty(\mathfrak{m})$. It is sufficient to prove the thesis for a function g satisfying $g \geq c > 0$. For any $n \in \mathbb{N}$ and $i \in \mathbb{Z}$, let us denote $I_{ni} := \big[\frac{i}{n}, \frac{i+1}{n}\big[$. Call φ_{ni} the continuous function that is the identity on I_{ni} and constant elsewhere. Let us define $f_{ni} := f - \frac{i-1}{n}$ and $\tilde{f}_{ni} := \varphi_{ni} \circ f - \frac{i-1}{n}$. Notice that $f_{ni} = \tilde{f}_{ni}$ holds $\mathfrak{m}$-a.e. in $f^{-1}(I_{ni})$, whence $|Df_{ni}| = |D\tilde{f}_{ni}| = |Df|$ and $\big|D(f_{ni}\, g)\big| = \big|D(\tilde{f}_{ni}\, g)\big|$ are verified $\mathfrak{m}$-a.e. in $f^{-1}(I_{ni})$ by locality. Moreover, we have that $1/n \leq \tilde{f}_{ni} \leq 2/n$ holds $\mathfrak{m}$-a.e. in X. Therefore

$$\begin{aligned} |D(fg)| &\leq \big|D(f_{ni}\, g)\big| + \frac{|i-1|}{n}\, |Dg| \leq |g||D\tilde{f}_{ni}| + |\tilde{f}_{ni}||Dg| + \frac{|i-1|}{n}\, |Dg| \\ &\leq |g||Df| + |Dg| \left(|f| + \frac{4}{n}\right) \qquad \mathfrak{m}\text{-a.e. in } f^{-1}(I_{ni}), \end{aligned}$$

where the second inequality follows from the case $f, g \geq c > 0$ treated above. This implies that the inequality $|D(fg)| \leq |f||Dg| + |g||Df| + 4\,|Dg|/n$ holds $\mathfrak{m}$-a.e. in X. Since $n \in \mathbb{N}$ is arbitrary, the Leibniz rule follows. ■

Remark 8.9 Property C) of Theorem 8.7 can be easily seen to hold for every $f \in W^{1,2}(\mathrm{X})$ and $g \in \mathrm{LIP}_b(\mathrm{X})$. ■

We can now introduce the *local Sobolev class* associated to $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$:

Definition 8.10 *We define $\mathrm{S}^2_{loc}(\mathrm{X})$ as the set of all Borel functions $f : \mathrm{X} \to \mathbb{R}$ with the following property: for any bounded Borel set $B \subseteq \mathrm{X}$, there exists a function $f_B \in \mathrm{S}^2(\mathrm{X})$ such that $f_B = f$ holds $\mathfrak{m}$-a.e. in B. Given any $f \in \mathrm{S}^2_{loc}(\mathrm{X})$, we define the function $|Df|$ as*

$$|Df| := |Df_B| \quad \mathfrak{m}\text{-a.e. in } B, \qquad \begin{array}{l}\text{for any bounded Borel set } B \subseteq \mathrm{X} \text{ and for}\\ \text{any } f_B \in \mathrm{S}^2(\mathrm{X}) \text{ with } f_B = f \;\; \mathfrak{m}\text{-a.e. in } B.\end{array} \tag{8.7}$$

The well-posedness of definition (8.7) stems from the locality property of minimal weak upper gradients, which had been proved in Theorem 8.7.

We define $L^2_{loc}(\mathrm{X})$ as the space of all Borel functions $g : \mathrm{X} \to \mathbb{R}$ such that $g_{|B} \in L^2(\mathfrak{m})$ for every bounded Borel subset B of X. It is then clear that $|Df| \in L^2_{loc}(\mathrm{X})$ for any $f \in \mathrm{S}^2_{loc}(\mathrm{X})$.

Proposition 8.11 (Alternative characterisation of $\mathrm{S}^2_{loc}(\mathrm{X})$, pt. 1) *Let $f \in \mathrm{S}^2_{loc}(\mathrm{X})$ be given. Then it holds that*

$$\int \big|f(\gamma_1) - f(\gamma_0)\big| \, \mathrm{d}\boldsymbol{\pi}(\gamma) \leq \iint_0^1 |Df|(\gamma_t)|\dot{\gamma}_t| \, \mathrm{d}t \, \mathrm{d}\boldsymbol{\pi}(\gamma) \qquad \textit{for every } \boldsymbol{\pi} \textit{ test plan.} \tag{8.8}$$

Proof. Fix a test plan $\boldsymbol{\pi}$ and a point $\bar{x} \in \mathrm{X}$. For any $n \in \mathbb{N}$, let us define

$$\Gamma_n := \left\{ \gamma : [0,1] \to \mathrm{X} \;\; \mathrm{AC} \;\middle|\; \mathsf{d}(\gamma_0, \bar{x}) \leq n \text{ and } \int_0^1 |\dot{\gamma}_t|^2 \, \mathrm{d}t \leq n \right\},$$

which turns out to be a closed subset of $C([0,1], \mathrm{X})$. It is clear that $\boldsymbol{\pi}\big(\bigcup_n \Gamma_n\big) = 1$. Now let us call $\boldsymbol{\pi}_n := \boldsymbol{\pi}(\Gamma_n)^{-1} \, \boldsymbol{\pi}_{|\Gamma_n}$ for every $n \in \mathbb{N}$ such that $\boldsymbol{\pi}(\Gamma_n) > 0$. For $\boldsymbol{\pi}_n$-a.e. γ it holds that

$$\mathsf{d}(\gamma_t, \bar{x}) \leq \int_0^t |\dot{\gamma}_s| \, \mathrm{d}s + \mathsf{d}(\gamma_0, \bar{x}) \leq \left(\int_0^1 |\dot{\gamma}_s|^2 \, \mathrm{d}s \right)^{1/2} + n \leq \sqrt{n} + n \qquad \text{for every } t \in [0,1].$$

Denote by B_n the open ball of radius $\sqrt{n}+n+1$ centered at $\bar{x}$ and take any function $f_n \in \mathrm{S}^2(\mathrm{X})$ such that $f_n = f$ holds $\mathfrak{m}$-a.e. in B_n. Therefore for $\boldsymbol{\pi}_n$-a.e. curve γ one has that

$$\big|f(\gamma_1) - f(\gamma_0)\big| = \big|f_n(\gamma_1) - f_n(\gamma_0)\big| \leq \int_0^1 |Df_n|(\gamma_t)|\dot{\gamma}_t| \, \mathrm{d}t = \int_0^1 |Df|(\gamma_t)|\dot{\gamma}_t| \, \mathrm{d}t,$$

whence (8.8) follows by arbitrariness of n. □

9 Lesson [08/11/2017]

Given a Polish space X and a (signed) Borel measure μ on X, we define the *support* of μ as

$$\mathrm{spt}(\mu) := \bigcap \big\{ C \subseteq \mathrm{X} \text{ closed } : \mu^+(\mathrm{X} \setminus C) = \mu^-(\mathrm{X} \setminus C) = 0 \big\}. \tag{9.1}$$

Clearly $\mathrm{spt}(\mu)$ is a closed subset of X by construction.

Remark 9.1 We point out that

$$\mu|_{\mathrm{X} \setminus \mathrm{spt}(\mu)} = 0. \tag{9.2}$$

Indeed, since X is a Lindelöf space (as it is separable), we can choose a sequence $(U_n)_n$ of open sets such that $\bigcup_n U_n = \bigcup \big\{ \mathrm{X} \setminus C : C \text{ closed}, |\mu|(\mathrm{X} \setminus C) = 0 \big\}$, whence

$$|\mu|\big(\mathrm{X} \setminus \mathrm{spt}(\mu)\big) = |\mu|\Big(\bigcup_n U_n \Big) \leq \sum_n |\mu|(U_n) = 0,$$

which is equivalent to (9.2). ■

We can prove the converse of Proposition 8.11 under the additional assumption that the function f belongs to the space $L^2_{loc}(\mathrm{X})$.

Proposition 9.2 (Alternative characterisation of $\mathrm{S}^2_{loc}(\mathrm{X})$, pt. 2) *Let $f \in L^2_{loc}(\mathrm{X})$ be a given map. Suppose that $G \in L^2_{loc}(\mathrm{X})$ is a non-negative function satisfying*

$$\int \big| f(\gamma_1) - f(\gamma_0) \big| \,\mathrm{d}\boldsymbol{\pi}(\gamma) \leq \iint_0^1 G(\gamma_t)|\dot{\gamma}_t| \,\mathrm{d}t \,\mathrm{d}\boldsymbol{\pi}(\gamma) \quad \textit{for every } \boldsymbol{\pi} \textit{ test plan.} \tag{9.3}$$

Then $f \in \mathrm{S}^2_{loc}(\mathrm{X})$ and $|Df| \leq G$ holds $\mathfrak{m}$-a.e. in X.

Proof. Step 1. We say that a test plan $\boldsymbol{\pi}$ is bounded provided $\big\{ \gamma_t : \gamma \in \mathrm{spt}(\boldsymbol{\pi}), t \in [0,1] \big\}$ is bounded. By arguing as in the proof of Theorem 7.7, one can prove the following claim:

Fix $f : \mathrm{X} \to \mathbb{R}$ Borel, $\boldsymbol{\pi}$ bounded test plan and $G \in L^2_{loc}(\mathrm{X})$ with $G \geq 0$. TFAE:

A) (9.3) holds for every test plan $\boldsymbol{\pi}'$ of the form $(\mathsf{Restr}_s^t)_*\big(\boldsymbol{\pi}(\Gamma)^{-1}\,\boldsymbol{\pi}|_{\Gamma}\big)$, (9.4)

B) for $\boldsymbol{\pi}$-a.e. γ we have $f \circ \gamma \in W^{1,1}(0,1)$ and $\big|(f \circ \gamma)'_t\big| \leq G(\gamma_t)|\dot{\gamma}_t|$ for a.e. t.

Step 2. Fix a function $f \in L^2_{loc}(\mathrm{X})$ satisfying (9.3), a test plan $\boldsymbol{\pi}$ on X and a Lipschitz function $g \in \mathrm{LIP}_{bs}(\mathrm{X})$. Given $\bar{x} \in \mathrm{X}$ and $n \in \mathbb{N}$, let us define

$$\Gamma_n := \left\{ \gamma : [0,1] \to \mathrm{X} \ \text{ AC} \ \middle| \ \mathsf{d}(\gamma_0, \bar{x}) \leq n \text{ and } \int_0^1 |\dot{\gamma}_t|^2 \,\mathrm{d}t \leq n \right\},$$

so that each Γ_n is a Borel set and $\boldsymbol{\pi}\big(\bigcup_n \Gamma_n\big) = 1$, as in the proof of Proposition 8.11. Let us fix $n \in \mathbb{N}$ sufficiently big and define $\boldsymbol{\pi}_n := \boldsymbol{\pi}(\Gamma_n)^{-1}\,\boldsymbol{\pi}|_{\Gamma_n}$, so that $\boldsymbol{\pi}_n$ is a bounded test plan on X. Now choose any open bounded set Ω containing $\mathrm{spt}(g)$, whence we have that the inequality $\big|(g \circ \gamma)'_t\big| \leq |Dg|\,|\dot{\gamma}_t|\,\chi_\Omega(\gamma_t)$ holds for $(\boldsymbol{\pi}_n \times \mathcal{L}_1)$-a.e. (γ, t). Thus B) of (9.4) gives

$$\big|\big((fg) \circ \gamma\big)'_t\big| \leq |f|(\gamma_t)\,\big|(g \circ \gamma)'_t\big| + |g|(\gamma_t)\,\big|(f \circ \gamma)'_t\big| \leq \big(\chi_\Omega\,|g|\,G + \chi_\Omega\,|f|\,|Dg|\big)(\gamma_t)\,|\dot{\gamma}_t|$$

for $(\boldsymbol{\pi}_n \times \mathcal{L}_1)$-a.e. (γ, t), so also for $(\boldsymbol{\pi} \times \mathcal{L}_1)$-a.e. (γ, t). Note that $\chi_\Omega\big(|g|G + |f||Dg|\big) \in L^2(\mathfrak{m})$. Therefore Theorem 7.7 grants that $fg \in \mathrm{S}^2(\mathrm{X})$ and $|D(fg)| \leq \chi_\Omega\big(|g|G + |f||Dg|\big)$.
Step 3. To conclude, fix $f \in L^2_{loc}(\mathrm{X})$ satisfying (9.3). Given a bounded Borel set $B \subseteq \mathrm{X}$, pick a function $g \in \mathrm{LIP}_{bs}(\mathrm{X})$ with $g = 1$ on B, thus $|Dg| = 0$ holds $\mathfrak{m}$-a.e. in B by locality. Hence Step 2 implies that $|Df| = |D(fg)| \leq G$ holds $\mathfrak{m}$-a.e. in B, yielding the thesis. □

Corollary 9.3 *Let $f : \mathrm{X} \to \mathbb{R}$ be a Borel map. Then $f \in \mathrm{S}^2(\mathrm{X})$ if and only if $f \in \mathrm{S}^2_{loc}(\mathrm{X})$ and $|Df| \in L^2(\mathfrak{m})$.*

Proof. Immediate consequence of Proposition 8.11 and Proposition 9.2. □

We now aim to prove that the definition of Sobolev space for abstract metric measure spaces is consistent with the classical one when we work in the Euclidean setting, namely if we consider $(\mathrm{X}, \mathsf{d}, \mathfrak{m}) = (\mathbb{R}^n, \mathsf{d}_{\mathrm{Eucl}}, \mathcal{L}^n)$. To this purpose, let us fix some notation:

$$\begin{aligned} W^{1,2}(\mathbb{R}^n) &= \text{ the classical Sobolev space on } \mathbb{R}^n, \\ |Df| &= \text{ the minimal weak upper gradient of } f \in \mathrm{S}^2_{loc}(\mathbb{R}^n), \\ \mathrm{d}f &= \text{ the distributional differential of } f \in W^{1,2}_{loc}(\mathbb{R}^n), \\ \nabla f &= \text{ the `true' gradient of } f \in C^\infty(\mathbb{R}^n). \end{aligned}$$

The above-mentioned consistency can be readily got as a consequence of the following facts:

Proposition 9.4 *The following hold:*

A) *If $f \in C^\infty(\mathbb{R}^n) \subseteq W^{1,2}_{loc}(\mathbb{R}^n)$, then the function f belongs to the space $\mathrm{S}^2_{loc} \cap L^2_{loc}(\mathbb{R}^n)$ and the equalities $|\nabla f| = |\mathrm{d}f| = |Df|$ hold $\mathcal{L}^n$-a.e. in $\mathbb{R}^n$.*

B) *If $f \in W^{1,2}(\mathbb{R}^n)$ and $\rho \in C^\infty_c(\mathbb{R}^n)$ is a convolution kernel, then $f * \rho \in W^{1,2}(\mathbb{R}^n)$ and the inequality $\big|\mathrm{d}(f * \rho)\big| \leq |\mathrm{d}f| * \rho$ holds $\mathcal{L}^n$-a.e. in $\mathbb{R}^n$.*

C) *If $f \in \mathrm{S}^2 \cap L^2(\mathbb{R}^n)$ and $\rho \in C^\infty_c(\mathbb{R}^n)$ is a convolution kernel, then $f * \rho \in \mathrm{S}^2 \cap L^2(\mathbb{R}^n)$ and the inequality $\big|D(f * \rho)\big| \leq |Df| * \rho$ holds $\mathcal{L}^n$-a.e. in $\mathbb{R}^n$.*

Proof. A) It is well-known that $|\nabla f| = |\mathrm{d}f|$ holds $\mathcal{L}^n$-a.e.. Moreover, $|Df| \leq \mathrm{lip}(f) = |\nabla f|$ is satisfied $\mathcal{L}^n$-a.e., thus to conclude it suffices to show that $\int |Df| \,\mathrm{d}\mathcal{L}^n \geq \int |\nabla f| \,\mathrm{d}\mathcal{L}^n$. By monotone convergence theorem, it is enough to prove that $\int_K |Df| \,\mathrm{d}\mathcal{L}^n \geq \int_K |\nabla f| \,\mathrm{d}\mathcal{L}^n$ is satisfied for any compact subset K of the open set $\big\{|\nabla f| > 0\big\}$. Then let us fix such a compact set K and some $\varepsilon > 0$. Call $\lambda := \min_K |\nabla f| > 0$. We can take a Borel partition $(U_i)_{i=1}^k$ of K and vectors $(v_i)_{i=1}^k \subseteq \mathbb{R}^n$ such that $\mathcal{L}^n(U_i) > 0$, $|v_i| \geq \lambda$ and $\big|\nabla f(x) - v_i\big| < \varepsilon$ for every $x \in U_i$. Fix $i = 1, \ldots, k$. Call $\mu := \mathcal{L}^n(U_i)^{-1} \mathcal{L}^n|_{U_i}$ and $\boldsymbol{\pi} := F_* \mu$, where $F : \mathbb{R}^n \to C([0,1], \mathbb{R}^n)$ is given by $x \mapsto \big(t \mapsto x + t v_i\big)$, so that $(\mathrm{e}_t)_* \boldsymbol{\pi} \leq \mathcal{L}^n(U_i)^{-1} (\cdot + t v_i)_* \mathcal{L}^n \leq \mathcal{L}^n(U_i)^{-1} \mathcal{L}^n$ holds for every $t \in [0,1]$ and $\iint_0^1 |\dot\gamma_t|^2 \,\mathrm{d}t \,\mathrm{d}\boldsymbol{\pi}(\gamma) = |v_i|^2 < +\infty$, which means that $\boldsymbol{\pi}$ is a test plan on $\mathbb{R}^n$.

It is clear that $f \in \mathrm{S}^2_{loc} \cap L^2_{loc}(\mathbb{R}^n)$, whence for any $t \in [0,1]$ one has

$$\begin{aligned}\int \big|f(\gamma_t) - f(\gamma_0)\big| \,\mathrm{d}\boldsymbol{\pi}(\gamma) &\leq \iint_0^t |Df|(\gamma_s)|\dot{\gamma}_s| \,\mathrm{d}s \,\mathrm{d}\boldsymbol{\pi}(\gamma) = |v_i| \iint_0^t |Df|(\gamma_s) \,\mathrm{d}s \,\mathrm{d}\boldsymbol{\pi}(\gamma)\\ &= |v_i| \int_0^t\!\!\int |Df| \,\mathrm{d}(\mathrm{e}_s)_*\boldsymbol{\pi} \,\mathrm{d}s = |v_i| \int_0^t\!\!\int |Df| \,\mathrm{d}(\cdot + s v_i)_*\mu \,\mathrm{d}s\\ &= \frac{|v_i|}{\mathcal{L}^n(U_i)} \int_0^t\!\!\int \chi_{U_i + s v_i} \,|Df| \,\mathrm{d}\mathcal{L}^n \,\mathrm{d}s.\end{aligned}$$

Since $\chi_{U_i+sv_i}$ converges to χ_{U_i} in $L^2(\mathbb{R}^n)$ as $s \to 0$, if we divide the previous formula by t and we let $t \searrow 0$, then we obtain that

$$\begin{aligned}|v_i| \fint_{U_i} |Df| \,\mathrm{d}\mathcal{L}^n &\geq \int \big|\langle \nabla f(\gamma_0), \gamma_0' \rangle\big| \,\mathrm{d}\boldsymbol{\pi}(\gamma) = \int \big|\langle \nabla f, v_i \rangle\big| \,\mathrm{d}(\mathrm{e}_0)_*\boldsymbol{\pi} = \fint_{U_i} \big|\langle \nabla f, v_i \rangle\big| \,\mathrm{d}\mathcal{L}^n\\ &\geq \big(|v_i| - 2\,\varepsilon\big) \fint_{U_i} |\nabla f| \,\mathrm{d}\mathcal{L}^n,\end{aligned}$$

where the last inequality follows from $\big|\langle \nabla f, v_i \rangle\big| \geq |\nabla f||v_i| - 2\,|\nabla f||\nabla f - v_i|$. Therefore

$$\begin{aligned}\int_K |Df| \,\mathrm{d}\mathcal{L}^n &= \sum_{i=1}^k \mathcal{L}^n(U_i) \fint_{U_i} |Df| \,\mathrm{d}\mathcal{L}^n \geq \sum_{i=1}^k \mathcal{L}^n(U_i) \left[\fint_{U_i} |\nabla f| \,\mathrm{d}\mathcal{L}^n - \frac{2\,\varepsilon}{|v_i|} \fint_{U_i} |\nabla f| \,\mathrm{d}\mathcal{L}^n\right]\\ &\geq \int_K |\nabla f| \,\mathrm{d}\mathcal{L}^n - \frac{2\,\varepsilon}{\lambda} \int_K |\nabla f| \,\mathrm{d}\mathcal{L}^n.\end{aligned}$$

By letting $\varepsilon \searrow 0$ we thus conclude that $\int_K |Df| \,\mathrm{d}\mathcal{L}^n \geq \int_K |\nabla f| \,\mathrm{d}\mathcal{L}^n$, as required.

B) It is well-known that $f * \rho \in W^{1,2}(\mathbb{R}^n)$ and $\mathrm{d}(f * \rho) = (\mathrm{d}f) * \rho$. To conclude, it only remains to observe that $\big|(\mathrm{d}f) * \rho\big| \leq |\mathrm{d}f| * \rho$. Hence property B) is achieved.

C) Given any $x \in \mathbb{R}^n$, let us define $\mathsf{Tr}_x : C([0,1],\mathbb{R}^n) \to C([0,1],\mathbb{R}^n)$ as $\mathsf{Tr}_x(\gamma)_t := \gamma_t - x$. If γ is absolutely continuous, then γ and $\mathsf{Tr}_x(\gamma)$ have the same metric speed. Now fix a test plan $\boldsymbol{\pi}$. Clearly $(\mathsf{Tr}_x)_*\boldsymbol{\pi}$ is a test plan as well. Therefore

$$\begin{aligned}\int \big|(f*\rho)(\gamma_1) - (f*\rho)(\gamma_0)\big| \,\mathrm{d}\boldsymbol{\pi}(\gamma) &\leq \int \rho(x) \int \big|f(\gamma_1 - x) - f(\gamma_0 - x)\big| \,\mathrm{d}\boldsymbol{\pi}(\gamma) \,\mathrm{d}x\\ &= \int \rho(x) \int \big|f(\sigma_1) - f(\sigma_0)\big| \,\mathrm{d}(\mathsf{Tr}_x)_*\boldsymbol{\pi}(\sigma) \,\mathrm{d}x\\ &\leq \int \rho(x) \iint_0^1 |Df|(\sigma_t) \,|\dot{\sigma}_t| \,\mathrm{d}t \,\mathrm{d}(\mathsf{Tr}_x)_*\boldsymbol{\pi}(\sigma) \,\mathrm{d}x\\ &= \iiint_0^1 \rho(x) \,|Df|(\gamma_t - x) \,|\dot{\gamma}_t| \,\mathrm{d}t \,\mathrm{d}\boldsymbol{\pi}(\gamma) \,\mathrm{d}x\\ &= \iint_0^1 \left(\int |Df|(\gamma_t - x) \,\rho(x) \,\mathrm{d}x\right) |\dot{\gamma}_t| \,\mathrm{d}t \,\mathrm{d}\boldsymbol{\pi}(\gamma)\\ &= \iint_0^1 \big(|Df| * \rho\big)(\gamma_t) \,|\dot{\gamma}_t| \,\mathrm{d}t \,\mathrm{d}\boldsymbol{\pi}(\gamma),\end{aligned}$$

which grants that $f * \rho \in \mathrm{S}^2 \cap L^2(\mathbb{R}^n)$ and $\big|D(f*\rho)\big| \leq |Df| * \rho$ holds $\mathcal{L}^n$-a.e. in $\mathbb{R}^n$. $\square$

We are now in a position to prove the main result:

Theorem 9.5 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a given Borel function. Then $f \in \mathrm{S}^2 \cap L^2(\mathbb{R}^n)$ if and only if $f \in W^{1,2}(\mathbb{R}^n)$. In this case, the equality $|Df| = |\mathrm{d}f|$ holds $\mathcal{L}^n$-a.e. in $\mathbb{R}^n$.*

Proof. Let us fix a family of convolution kernels $(\rho_\varepsilon)_{\varepsilon>0}$. Given any $f \in W^{1,2}(\mathbb{R}^n)$, we deduce from properties A) and B) of Proposition 9.4 that $f * \rho_\varepsilon \in \mathrm{S}^2 \cap L^2(\mathbb{R}^n)$ and that

$$\big|D(f*\rho_\varepsilon)\big| = \big|\mathrm{d}(f*\rho_\varepsilon)\big| \leq |\mathrm{d}f| * \rho_\varepsilon \longrightarrow |\mathrm{d}f| \quad \text{in } L^2(\mathbb{R}^n) \qquad \text{as } \varepsilon \searrow 0.$$

Since also $f * \rho_\varepsilon \to f$ in $L^2(\mathbb{R}^n)$ as $\varepsilon \searrow 0$, we have that $f \in \mathrm{S}^2 \cap L^2(\mathbb{R}^n)$ and that $|Df| \leq |\mathrm{d}f|$ holds $\mathcal{L}^n$-a.e. in $\mathbb{R}^n$, as a consequence of Proposition 4.11.

On the other hand, given any function $f \in \mathrm{S}^2 \cap L^2(\mathbb{R}^n)$, we have that $f * \rho_\varepsilon \in \mathrm{S}^2 \cap L^2(\mathbb{R}^n)$ and that $\big|\mathrm{d}(f*\rho_\varepsilon)\big| = \big|D(f*\rho_\varepsilon)\big| \leq |Df| * \rho_\varepsilon$ holds $\mathcal{L}^n$-a.e. by properties A) and C) of Proposition 9.4. Since $|Df| * \rho_\varepsilon \to |Df|$ in $L^2(\mathbb{R}^n)$ as $\varepsilon \searrow 0$, there exist a sequence $\varepsilon_k \searrow 0$ and $w \in L^2(\mathbb{R}^n)$ such that $\mathrm{d}(f*\rho_{\varepsilon_k}) \rightharpoonup w$ weakly in $L^2(\mathbb{R}^n)$, thus necessarily $w = \mathrm{d}f$. In particular, it holds that $\int |\mathrm{d}f|^2 \,\mathrm{d}\mathcal{L}^n \leq \underline{\lim}_k \int \big|\mathrm{d}(f*\rho_{\varepsilon_k})\big|^2 \,\mathrm{d}\mathcal{L}^n = \int |Df|^2 \,\mathrm{d}\mathcal{L}^n$, which forces the $\mathcal{L}^n$-a.e. equality $|Df| = |\mathrm{d}f|$, proving the thesis. $\square$

Let us come back to the case of a generic metric measure space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$. We want to prove that the Sobolev space $W^{1,2}(\mathrm{X})$ is separable whenever it is reflexive. To do it, we need the following result, of purely functional analytic nature:

Lemma 9.6 *Let $\mathbb{E}_1, \mathbb{E}_2$ be Banach spaces. Let $i : \mathbb{E}_1 \to \mathbb{E}_2$ be a linear and continuous injection. Suppose that $\mathbb{E}_1$ is reflexive and that $\mathbb{E}_2$ is separable. Then $\mathbb{E}_1$ is separable as well.*

Proof. Recall that any continuous bijection f from a compact topological space X to a Hausdorff topological space Y is a homeomorphism (each closed subset $C \subseteq \mathrm{X}$ is compact because X is compact, hence $f(C)$, being compact in the Hausdorff space Y, is closed). Call

- X the closed unit ball in $\mathbb{E}_1$ endowed with the (restriction of the) weak topology of $\mathbb{E}_1$,
- Y the image $i(\mathrm{X})$ endowed with the (restriction of the) weak topology of $\mathbb{E}_2$,
- f the map $i_{|\mathrm{X}}$ from X to Y.

Since X is compact (by reflexivity of $\mathbb{E}_1$), Y is Hausdorff and f is continuous (as i is linear and continuous), we thus deduce that f is a homeomorphism. In particular, the separability of Y grants that X is separable as well, i.e. the closed unit ball B of $\mathbb{E}_1$ is weakly separable. Now fix a countable weakly dense subset D of such ball. Denote by Q the set of all finite convex combinations with coefficients in $\mathbb{Q}$ of elements of D. It is clear that the set Q, which is countable by construction, is strongly dense in the convex hull C of D. Since C is convex, we have that the weak closure and the strong closure of C coincide. Moreover, such closure contains B. Hence Q is strongly dense in the set B, which accordingly turns out to be strongly separable. Finally, we conclude that $\mathbb{E}_1 = \bigcup_{n\in\mathbb{N}} nB$ is strongly separable as well, thus achieving the thesis. $\square$

Hence we can immediately deduce from such lemma that

Theorem 9.7 *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a metric measure space. Suppose that* $W^{1,2}(\mathrm{X})$ *is reflexive. Then* $W^{1,2}(\mathrm{X})$ *is separable.*

Proof. Apply Lemma 9.6 to $\mathbb{E}_1 = W^{1,2}(\mathrm{X})$, $\mathbb{E}_2 = L^2(\mathfrak{m})$ and i the inclusion $\mathbb{E}_1 \hookrightarrow \mathbb{E}_2$. □

10 Lesson [27/11/2017]

We start by stating and proving a well-known functional analytic result, which will be needed in the forthcoming discussion:

Theorem 10.1 (Mazur's lemma) *Let* $\mathbb{B}$ *be a Banach space. Let* $(v_n)_n \subseteq \mathbb{B}$ *be a sequence that weakly converges to some limit* $v \in \mathbb{B}$. *Then there exist* $(N_n)_n \subseteq \mathbb{N}$ *and* $(\alpha_{n,i})_{i=n}^{N_n} \subseteq [0,1]$ *such that* $\sum_{i=n}^{N_n} \alpha_{n,i} = 1$ *for all* $n \in \mathbb{N}$ *and* $\tilde{v}_n := \sum_{i=n}^{N_n} \alpha_{n,i}\, v_i \to v$ *in the strong topology of* $\mathbb{B}$.

Proof. Given any $n \in \mathbb{N}$, let us denote by K_n the strong closure of the set of all (finite) convex combinations of the $(v_i)_{i\geq n}$. Each set K_n, being strongly closed and convex, is weakly closed by Hahn-Banach theorem. Given that $v \in \bigcap_{n\in\mathbb{N}} K_n$, for every $n \in \mathbb{N}$ we can choose $N_n \geq n$ and some $\alpha_{n,n},\ldots,\alpha_{n,N_n} \in [0,1]$ such that $\sum_{i=n}^{N_n} \alpha_{n,i} = 1$ and $\|\tilde{v}_n - v\|_{\mathbb{B}} < 1/n$, where we put $\tilde{v}_n := \sum_{i=n}^{N_n} \alpha_{n,i}\, v_i$. This proves the claim. □

We now introduce some alternative definitions of Sobolev space on a general metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m})$, which a posteriori turn out to be equivalent to the one (via weak upper gradients) we gave in Definition 4.14. Roughly speaking, what we need is an $L^2(\mathfrak{m})$-lsc energy functional of the form $\frac{1}{2}\int |\mathrm{d}f|^2\,\mathrm{d}\mathfrak{m}$, where the function $|\mathrm{d}f|$ is an object which is 'local' and satisfies some sort of chain rule. Given any Lipschitz function $f \in \mathrm{LIP}(\mathrm{X})$, some (seemingly) good candidates for $|\mathrm{d}f|$ could be given by

$$\begin{aligned} \mathrm{lip}(f)(x) &:= \varlimsup_{y\to x} \frac{\big|f(y)-f(x)\big|}{\mathsf{d}(y,x)} \qquad \text{(local Lipschitz constant)},\\ \mathrm{lip}_a(f)(x) &:= \varlimsup_{y,z\to x} \frac{\big|f(y)-f(z)\big|}{\mathsf{d}(y,z)} \qquad \text{(asymptotic Lipschitz constant)}, \end{aligned}$$

for $x \in \mathrm{X}$ accumulation point and $\mathrm{lip}(f)(x), \mathrm{lip}_a(f)(x) := 0$ otherwise. The local Lipschitz constant had been previously introduced in (8.1). Observe that $\mathrm{lip}(f) \leq \mathrm{lip}_a(f) \leq \mathrm{Lip}(f)$ and that the equalities $\mathrm{lip}_a(f)(x) = \lim_{r\searrow 0} \mathrm{Lip}\big(f_{|B_r(x)}\big) = \inf_{r>0} \mathrm{Lip}\big(f_{|B_r(x)}\big)$ hold for every accumulation point $x \in \mathrm{X}$. Moreover, we shall make use of the following property of lip_a:

$$\mathrm{lip}_a(fg) \leq |f|\,\mathrm{lip}_a(g) + |g|\,\mathrm{lip}_a(f) \qquad \text{for every } f,g \in \mathrm{LIP}(\mathrm{X}), \tag{10.1}$$

which is the *Leibniz rule* for the asymptotic Lipschitz constant.

Exercise 10.2 Prove that $\mathrm{lip}_a(f)$ is an upper semicontinuous function.

Another ingredient that we need is the notion of upper gradient, which has been already introduced in Definition 8.5. For the sake of convenience, we restate it here:

Definition 10.3 (Upper gradient) *Let $f : \mathrm{X} \to \mathbb{R}$ be a given continuous function. Then a Borel function $G : \mathrm{X} \to [0, +\infty]$ is said to be an* upper gradient *of f provided*

$$\big|f(\gamma_1) - f(\gamma_0)\big| \leq \int_0^1 G(\gamma_t)|\dot{\gamma}_t| \,\mathrm{d}t \qquad \textit{holds for every AC curve } \gamma. \tag{10.2}$$

Given any Lipschitz function $f \in \mathrm{LIP}(\mathrm{X})$, it can be easily seen that $\mathrm{lip}(f)$, thus accordingly also $\mathrm{lip}_a(f)$, is an upper gradient of f.

Since, in general, the functionals $f \mapsto \frac{1}{2}\int \mathrm{lip}^2(f)\,\mathrm{d}\mathfrak{m}$ and $f \mapsto \frac{1}{2}\int \mathrm{lip}_a^2(f)\,\mathrm{d}\mathfrak{m}$ are not lsc, we have to introduce our energy functionals by means of a relaxation procedure:

Definition 10.4 *Let us give the following definitions:*

i) *The functional $\mathsf{E}_{*,a} : L^2(\mathfrak{m}) \to [0, +\infty]$ is given by*

$$\mathsf{E}_{*,a}(f) := \inf \varliminf_{n\to\infty} \frac{1}{2} \int \mathrm{lip}_a^2(f_n)\,\mathrm{d}\mathfrak{m},$$

where the infimum is taken among all sequences $(f_n)_n \subseteq \mathrm{LIP}(\mathrm{X})$ with $f_n \to f$ in $L^2(\mathfrak{m})$.

ii) *The functional $\mathsf{E}_* : L^2(\mathfrak{m}) \to [0, +\infty]$ is given by*

$$\mathsf{E}_*(f) := \inf \varliminf_{n\to\infty} \frac{1}{2} \int \mathrm{lip}^2(f_n)\,\mathrm{d}\mathfrak{m},$$

where the infimum is taken among all sequences $(f_n)_n \subseteq \mathrm{LIP}(\mathrm{X})$ with $f_n \to f$ in $L^2(\mathfrak{m})$.

iii) *The functional $\mathsf{E}_{\mathrm{Ch}} : L^2(\mathfrak{m}) \to [0, +\infty]$ is given by*

$$\mathsf{E}_{\mathrm{Ch}}(f) := \inf \varliminf_{n\to\infty} \frac{1}{2} \int G_n^2\,\mathrm{d}\mathfrak{m},$$

where the infimum is taken among all sequences $(f_n)_n \subseteq C(\mathrm{X})$ and $(G_n)_n$ such that G_n is an upper gradient of f_n for every $n \in \mathbb{N}$ and $f_n \to f$ in $L^2(\mathfrak{m})$.

Exercise 10.5 Prove that $\mathsf{E}_{*,a}$ is $L^2(\mathfrak{m})$-lower semicontinuous and is the maximal $L^2(\mathfrak{m})$-lsc functional E such that $\mathsf{E}(f) \leq \frac{1}{2}\int \mathrm{lip}_a^2(f)\,\mathrm{d}\mathfrak{m}$ holds for every $f \in \mathrm{LIP}(\mathrm{X})$. Actually, the same properties are verified by E_* if we replace $\mathrm{lip}_a(f)$ with $\mathrm{lip}(f)$.

Definition 10.6 *We define the Banach spaces $W^{1,2}_{*,a}(\mathrm{X})$, $W^{1,2}_*(\mathrm{X})$ and $W^{1,2}_{\mathrm{Ch}}(\mathrm{X})$ as follows:*

$$\begin{aligned} W^{1,2}_{*,a}(\mathrm{X}) &:= \big\{ f \in L^2(\mathfrak{m}) \,:\, \mathsf{E}_{*,a}(f) < +\infty \big\}, \\ W^{1,2}_{*}(\mathrm{X}) &:= \big\{ f \in L^2(\mathfrak{m}) \,:\, \mathsf{E}_{*}(f) < +\infty \big\}, \\ W^{1,2}_{\mathrm{Ch}}(\mathrm{X}) &:= \big\{ f \in L^2(\mathfrak{m}) \,:\, \mathsf{E}_{\mathrm{Ch}}(f) < +\infty \big\}. \end{aligned} \tag{10.3}$$

Any upper gradient is a weak upper gradient, thus $W^{1,2}_{*,a}(\mathrm{X}) \subseteq W^{1,2}_{*}(\mathrm{X}) \subseteq W^{1,2}_{\mathrm{Ch}}(\mathrm{X}) \subseteq W^{1,2}(\mathrm{X})$.

Hereafter, we shall mainly focus our attention on the space $W^{1,2}_{*,a}(\mathrm{X})$. Analogous statements for the other two spaces in (10.3) can be shown to hold.

Remark 10.7 The fact that the set $W^{1,2}_{*,a}(\mathrm{X})$ is a vector space follows from this observation: the asymptotic Lipschitz constant satisfies $\mathrm{lip}_a(f+g) \leq \mathrm{lip}_a(f)+\mathrm{lip}_a(g)$ for all $f,g \in \mathrm{LIP}(\mathrm{X})$. Given any $f,g \in W^{1,2}_{*,a}(\mathrm{X})$ and $\alpha,\beta \in \mathbb{R}$, we can choose two sequences $(f_n)_n,(g_n)_n \subseteq \mathrm{LIP}(\mathrm{X})$ such that $\lim_n \|f_n - f\|_{L^2(\mathfrak{m})} = \lim_n \|g_n - g\|_{L^2(\mathfrak{m})} = 0$ and $\overline{\lim}_n \int \mathrm{lip}_a^2(f_n) + \mathrm{lip}_a^2(g_n)\,\mathrm{d}\mathfrak{m}$ is finite. Since $\alpha f_n + \beta g_n \to \alpha f + \beta g$ in $L^2(\mathfrak{m})$, we thus deduce that

$$2\,\mathsf{E}_{*,a}(\alpha f + \beta g) \leq \varlimsup_n \int \mathrm{lip}_a^2(\alpha f_n + \beta g_n)\,\mathrm{d}\mathfrak{m} \leq 2 \varlimsup_n \int \alpha^2\,\mathrm{lip}_a^2(f_n) + \beta^2\,\mathrm{lip}_a^2(g_n)\,\mathrm{d}\mathfrak{m} < +\infty,$$

which shows that $\alpha f + \beta g \in W^{1,2}_{*,a}(\mathrm{X})$, as required. ∎

Definition 10.8 (Asymptotic relaxed slope) *Let $f \in W^{1,2}_{*,a}(\mathrm{X})$ be a given function. Then an element $G \in L^2(\mathfrak{m})$ with $G \geq 0$ is said to be an* asymptotic relaxed slope *for f provided there exists a sequence $(f_n)_n \subseteq \mathrm{LIP}(\mathrm{X})$ such that $f_n \to f$ strongly in $L^2(\mathfrak{m})$ and $\mathrm{lip}_a(f_n) \rightharpoonup G'$ weakly in $L^2(\mathfrak{m})$, for some $G' \in L^2(\mathfrak{m})$ with $G' \leq G$.*

Proposition 10.9 *Let $f \in W^{1,2}_{*,a}(\mathrm{X})$ be given. Then the set of all asymptotic relaxed slopes for f is a closed convex subset of $L^2(\mathfrak{m})$. Moreover, its element of minimal $L^2(\mathfrak{m})$-norm, denoted by $|Df|_{*,a}$ and called* minimal asymptotic relaxed slope*, satisfies the equality*

$$\mathsf{E}_{*,a}(f) = \frac{1}{2}\int |Df|^2_{*,a}\,\mathrm{d}\mathfrak{m}. \tag{10.4}$$

Proof. Convexity. Fix two asymptotic relaxed slopes G_1, G_2 for f and a constant $\alpha \in [0,1]$. For $i = 1,2$, choose $(f^i_n)_n \subseteq \mathrm{LIP}(\mathrm{X})$ such that $f^i_n \to f$ and $\mathrm{lip}_a(f^i_n) \rightharpoonup G'_i \leq G_i$. We then claim that $\alpha G_1 + (1-\alpha)G_2$ is an asymptotic relaxed slope for f. In order to prove it, notice that $\alpha f^1_n + (1-\alpha)f^2_n \to f$ in $L^2(\mathfrak{m})$ and that

$$\mathrm{lip}_a\big(\alpha f^1_n + (1-\alpha)f^2_n\big) \leq \alpha\,\mathrm{lip}_a(f^1_n) + (1-\alpha)\,\mathrm{lip}_a(f^2_n) \rightharpoonup \alpha G'_1 + (1-\alpha)G'_2 \leq \alpha G_1 + (1-\alpha)G_2.$$

Up to subsequence, we thus have that $\mathrm{lip}_a\big(\alpha f^1_n + (1-\alpha)f^2_n\big)$ weakly converges to some limit function $\widetilde{G} \leq \alpha G_1 + (1-\alpha)G_2$, proving the claim.

Closedness. Fix a sequence $(G_n)_n \subseteq L^2(\mathfrak{m})$ of asymptotic relaxed slopes for f that strongly converges to some $G \in L^2(\mathfrak{m})$. Given any $n \in \mathbb{N}$, we can pick a sequence $(f_{n,m})_m \subseteq \mathrm{LIP}(\mathrm{X})$ with $f_{n,m} \overset{m}{\to} f$ and $\mathrm{lip}_a(f_{n,m}) \overset{m}{\rightharpoonup} G'_n \leq G_n$. Up to subsequence, we have that $G'_n \to G'$ for some $G' \in L^2(\mathfrak{m})$ with $G' \leq G$. Then we can assume wlog that $\big(\mathrm{lip}_a(f_{n,m})\big)_{n,m}$ is bounded in the space $L^2(\mathfrak{m})$. Since the restriction of the weak topology to any closed ball of $L^2(\mathfrak{m})$ is metrizable, by a diagonalisation argument we can extract a subsequence $(m_n)_n$ for which we have $f_{n,m_n} \overset{n}{\to} f$ and $\mathrm{lip}_a(f_{n,m_n}) \overset{n}{\rightharpoonup} G' \leq G$, i.e. G is an asymptotic relaxed slope for f.

Formula (10.4). By a diagonalisation argument, there exists a sequence $(\tilde{f}_n)_n \subseteq \mathrm{LIP}(\mathrm{X})$ such that $\tilde{f}_n \to f$ in $L^2(\mathfrak{m})$ and $\mathsf{E}_{*,a}(f) = \lim_n \frac{1}{2}\int \mathrm{lip}_a^2(\tilde{f}_n)\,\mathrm{d}\mathfrak{m}$. Up to subsequence, it holds that $\operatorname{lip}_a(\tilde{f}_n) \rightharpoonup G$ for some $G \in L^2(\mathfrak{m})$. By Theorem 10.1, for any $n \in \mathbb{N}$ there exist $N_n \geq n$ and $(\alpha_{n,i})_{i=n}^{N_n} \subseteq [0,1]$ in such a way that $\sum_{i=n}^{N_n} \alpha_{n,i} = 1$ and $\sum_{i=n}^{N_n} \alpha_{n,i} \operatorname{lip}_a(\tilde{f}_i) \overset{n}{\to} G$ in $L^2(\mathfrak{m})$. Let us now define $f_n := \sum_{i=n}^{N_n} \alpha_{n,i} \tilde{f}_i$ for every $n \in \mathbb{N}$. It is clear that $f_n \to f$ in $L^2(\mathfrak{m})$: given any $\varepsilon > 0$, there exists $\bar{n} \in \mathbb{N}$ such that $\|\tilde{f}_n - f\|_{L^2(\mathfrak{m})} \leq \varepsilon$ for all $n \geq \bar{n}$, so that accordingly one has $\|f_n - f\|_{L^2(\mathfrak{m})} \leq \sum_{i=n}^{N_n} \alpha_{n,i} \|\tilde{f}_i - f\|_{L^2(\mathfrak{m})} \leq \varepsilon \sum_{i=n}^{N_n} \alpha_{n,i} = \varepsilon$ for every $n \geq \bar{n}$. Note that one has $\operatorname{lip}_a(f_n) \leq \sum_{i=n}^{N_n} \alpha_{n,i} \operatorname{lip}_a(\tilde{f}_i) \to G$ in $L^2(\mathfrak{m})$, whence (up to a not relabeled subsequence) it holds that $\operatorname{lip}_a(f_n) \rightharpoonup G' \leq G$. Therefore $\mathsf{E}_{*,a}(f) \leq \frac{1}{2} \int (G')^2 \, \mathrm{d}\mathfrak{m} \leq \frac{1}{2} \int G^2 \, \mathrm{d}\mathfrak{m} \leq \mathsf{E}_{*,a}(f)$, which forces $G' = G$ and $\operatorname{lip}_a(f_n) \to G$ in $L^2(\mathfrak{m})$. Hence $|Df|_{*,a} := G$ is the (unique) element of minimal $L^2(\mathfrak{m})$-norm in the family of all asymptotic relaxed slopes for the function f and the equality in (10.4) is satisfied. Then the thesis is achieved. □

Proposition 10.10 (Cheeger) *Let $f \in W^{1,2}_{*,a}(\mathrm{X})$ be given. Let G_1, G_2 be asymptotic relaxed slopes for f. Then $\min\{G_1, G_2\}$ is an asymptotic relaxed slope for f as well.*

Proof. Notice that $\min\{G_1, G_2\} = \chi_E \, G_1 + \chi_{E^c} \, G_2$, where $E := \{G_1 < G_2\}$. By inner regularity of the measure $\mathfrak{m}$, it thus suffices to show that $\chi_K \, G_1 + \chi_{K^c} \, G_2$ is an asymptotic relaxed slope for f, for any compact $K \subseteq \mathrm{X}$. Fix $r > 0$. Define the cut-off function $\eta_r \in L^2(\mathfrak{m})$ as $\eta_r := \big(1 - \mathsf{d}(\cdot, K)/r\big)^+$. For any $i = 1,2$, we can choose $(f^i_n)_n \subseteq \mathrm{LIP}(\mathrm{X})$ such that $f^i_n \to f$ and $\operatorname{lip}_a(f^i_n) \rightharpoonup G'_i \leq G_i$. Now call $h^r_n := \eta_r f^1_n + (1 - \eta_r) f^2_n \in \mathrm{LIP}(\mathrm{X})$ for every $n \in \mathbb{N}$. One clearly has that $h^r_n \overset{n}{\to} f$ strongly in $L^2(\mathfrak{m})$. Moreover, given that

$$h^r_n = f^1_n + (1 - \eta_r)(f^2_n - f^1_n) = f^2_n + \eta_r (f^1_n - f^2_n),$$

we infer from the Leibniz rule (10.1) that

$$\begin{aligned}
\operatorname{lip}_a(h^r_n) &\leq \operatorname{lip}_a(f^1_n) + (1 - \eta_r)\big(\operatorname{lip}_a(f^1_n) + \operatorname{lip}_a(f^2_n)\big) + |f^1_n - f^2_n| \operatorname{lip}_a(1 - \eta_r), \\
\operatorname{lip}_a(h^r_n) &\leq \operatorname{lip}_a(f^2_n) + \eta_r \big(\operatorname{lip}_a(f^1_n) + \operatorname{lip}_a(f^2_n)\big) + |f^1_n - f^2_n| \operatorname{lip}_a(\eta_r).
\end{aligned} \tag{10.5}$$

Up to subsequence, we obtain from (10.5) that $\operatorname{lip}_a(h^r_n) \overset{n}{\rightharpoonup} G_r$ for some $G_r \in L^2(\mathfrak{m})$ with

$$G_r \leq \min\big\{G'_1 + (1 - \eta_r)(G'_1 + G'_2), G'_2 + \eta_r (G'_1 + G'_2)\big\}. \tag{10.6}$$

Since $\eta_r = 1$ on K and $\eta_r = 0$ on $\mathrm{X} \setminus K^r$, where $K^r := \big\{x \in \mathrm{X} \,:\, \mathsf{d}(x, K) < r\big\}$, we deduce from the inequality (10.6) that

$$G_r \leq \chi_K \, G'_1 + \chi_{\mathrm{X} \setminus K^r} \, G'_2 + 2 \, \chi_{K^r \setminus K} \, (G'_1 + G'_2). \tag{10.7}$$

The right hand side in (10.7) converges in $L^2(\mathfrak{m})$ to the function $\chi_K \, G'_1 + \chi_{K^c} \, G'_2$ as $r \searrow 0$, which grants that $\chi_K \, G_1 + \chi_{K^c} \, G_2$ is an asymptotic relaxed slope for f, as required. □

It immediately follows from Proposition 10.10 that:

Corollary 10.11 *Let $f \in W^{1,2}_{*,a}(\mathrm{X})$. Take any asymptotic relaxed slope G for f. Then the inequality $|Df|_{*,a} \leq G$ holds $\mathfrak{m}$-a.e. in X.*

Proposition 10.12 (Chain rule) *Let $f \in W^{1,2}_{*,a}(\mathrm{X})$ be fixed. Let $\varphi \in C^1(\mathbb{R}) \cap \mathrm{LIP}(\mathbb{R})$ be such that $\varphi(0) = 0$, which grants that $\varphi \circ f \in L^2(\mathfrak{m})$. Then $\varphi \circ f \in W^{1,2}_{*,a}(\mathrm{X})$ and*

$$\big|D(\varphi \circ f)\big|_{*,a} \leq |\varphi'| \circ f \, |Df|_{*,a} \quad \textit{holds } \mathfrak{m}\textit{-a.e. in } \mathrm{X}. \tag{10.8}$$

Proof. Pick $(f_n)_n \subseteq \mathrm{LIP}(\mathrm{X})$ such that $f_n \to f$ and $\mathrm{lip}_a(f_n) \to |Df|_{*,a}$ in $L^2(\mathfrak{m})$. It holds that

$$\mathrm{lip}_a(\varphi \circ f_n) \leq |\varphi'| \circ f_n \, \mathrm{lip}_a(f_n) \longrightarrow |\varphi'| \circ f \, |Df|_{*,a} \quad \text{strongly in } L^2(\mathfrak{m}). \tag{10.9}$$

Then there exists $G \in L^2(\mathfrak{m})$ such that, possibly passing to a subsequence, $\mathrm{lip}_a(\varphi \circ f_n) \rightharpoonup G$. In particular $G \leq |\varphi'| \circ f \, |Df|_{*,a}$ by (10.9), while the inequality $\big|D(\varphi \circ f)\big|_{*,a} \leq G$ is granted by the minimality of $\big|D(\varphi \circ f)\big|_{*,a}$. This proves the thesis. □

Remark 10.13 Analogous properties to the ones that had been described in Theorem 8.7 can be shown to hold for the minimal asymptotic relaxed slope $|Df|_{*,a}$. This follows from Proposition 10.10 and Proposition 10.12 by suitably adapting the proof of Theorem 8.7. ■

The vector space $W^{1,2}_{*,a}(\mathrm{X})$ can be endowed with the norm

$$\|f\|^2_{W^{1,2}_{*,a}(\mathrm{X})} := \|f\|^2_{L^2(\mathfrak{m})} + \big\||Df|_{*,a}\big\|^2_{L^2(\mathfrak{m})} \quad \text{for every } f \in W^{1,2}_{*,a}(\mathrm{X}). \tag{10.10}$$

Then $\big(W^{1,2}_{*,a}(\mathrm{X}), \|\cdot\|_{W^{1,2}_{*,a}(\mathrm{X})}\big)$ turns out to be a Banach space. Completeness stems from the lower semicontinuity of the energy functional $\mathsf{E}_{*,a}$.

Remark 10.14 Similarly to what done so far, one can define the objects $|Df|_*$ and $|Df|_{\mathrm{Ch}}$. Then it holds that $|Df| \leq |Df|_{\mathrm{Ch}} \leq |Df|_* \leq |Df|_{*,a}$. ■

Besides the fact of granting completeness of $W^{1,2}_{*,a}(\mathrm{X})$, the relaxation procedure we used to define the energy functional $\mathsf{E}_{*,a}$ is also motivated by the following observation:

Remark 10.15 Suppose that X is compact. Define

$$\|f\|^2_{\widetilde{W}} := \|f\|^2_{L^2(\mathfrak{m})} + \big\|\mathrm{lip}_a(f)\big\|^2_{L^2(\mathfrak{m})} \quad \text{for every } f \in \mathrm{LIP}(\mathrm{X}).$$

Hence $\|\cdot\|_{\widetilde{W}}$ is a seminorm on the vector space LIP(X). Now let us denote by $\widetilde{W}$ the completion of the quotient space of $\big(\mathrm{LIP}(\mathrm{X}), \|\cdot\|_{\widetilde{W}}\big)$. The problem is that in general the elements of $\widetilde{W}$ *are not functions*, in the sense that we are going to explain. The natural inclusion $i : \mathrm{LIP}(\mathrm{X}) \to L^2(\mathfrak{m})$ uniquely extends to a linear continuous map $i : \widetilde{W} \to L^2(\mathfrak{m})$, but such map is not necessarily injective, as shown by the following example. ■

Example 10.16 Take $\mathrm{X} := [-1, 1]$ with the Euclidean distance and $\mathfrak{m} := \delta_0$. Consider the functions $f_1, f_2 \in \mathrm{LIP}(\mathrm{X})$ given by $f_1(x) := 0$ and $f_2(x) := x$, respectively. Then f_1 and f_2 coincide as elements of $L^2(\mathfrak{m})$, but $\|f_1 - f_2\|_{\widetilde{W}} = \|f_2\|_{\widetilde{W}} = 1$.

11 Lesson [29/11/2017]

We present a further notion of Sobolev space on metric measure spaces, which will turn out to be equivalent to all of the other ones discussed so far.

Given a metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m})$, let us define

$$\Gamma(\mathrm{X}) := \big\{\gamma : J \to \mathrm{X} \,\big|\, J \subseteq \mathbb{R} \text{ non-trivial interval, } \gamma \text{ AC}\big\}. \tag{11.1}$$

Given any curve $\gamma \in \Gamma(\mathrm{X})$, we will denote by $\mathrm{Dom}(\gamma)$ the interval where γ is defined and we will tipically call $I \in \mathbb{R}$ and $F \in \mathbb{R}$ the inf and the sup of $\mathrm{Dom}(\gamma)$, respectively.

If $G : \mathrm{X} \to [0,+\infty]$ is a Borel function and $\gamma \in \Gamma(\mathrm{X})$, then we define

$$\int_\gamma G := \int_I^F G(\gamma_t)|\dot\gamma_t|\,\mathrm{d}t, \tag{11.2}$$

with the convention that $\int_\gamma G := +\infty$ in the case in which $\big\{t \in \mathrm{Dom}(\gamma) : G(\gamma_t) = +\infty\big\}$ has positive $\mathcal{L}^1$-measure. We call $\int_\gamma G$ the *line integral* of G along the curve γ.

Definition 11.1 (2-modulus of a curve family) *Let Γ be any subset of $\Gamma(\mathrm{X})$. Then we define the quantity $\mathrm{Mod}_2(\Gamma) \in [0,+\infty]$ as*

$$\mathrm{Mod}_2(\Gamma) := \inf\left\{\int \rho^2\,\mathrm{d}\mathfrak{m} \,\middle|\, \rho : \mathrm{X} \to [0,+\infty] \textit{ Borel, } \int_\gamma \rho \geq 1 \textit{ for all } \gamma \in \Gamma\right\}. \tag{11.3}$$

We call $\mathrm{Mod}_2(\Gamma)$ *the* 2-modulus *of* Γ*. Moreover, a property is said to* hold 2-a.e. *provided it is satisfied for every γ belonging to some set $\Gamma \subseteq \Gamma(\mathrm{X})$ such that $\mathrm{Mod}_2(\Gamma^c) = 0$.*

The 2-modulus Mod_2 is an outer measure on $\Gamma(\mathrm{X})$, in particular it holds that

$$\begin{aligned} \Gamma \subseteq \Gamma' \subseteq \Gamma(\mathrm{X}) \quad &\Longrightarrow \quad \mathrm{Mod}_2(\Gamma) \leq \mathrm{Mod}_2(\Gamma'), \\ \Gamma_n \subseteq \Gamma(\mathrm{X}),\ \mathrm{Mod}_2(\Gamma_n) = 0 \text{ for all } n \in \mathbb{N} \quad &\Longrightarrow \quad \mathrm{Mod}_2(\Gamma) = 0, \text{ where } \Gamma := \bigcup_{n\in\mathbb{N}} \Gamma_n. \end{aligned}$$

To prove the above claim, fix a sequence $(\Gamma_n)_n$ of subsets of $\Gamma(\mathrm{X})$ and some constant $\varepsilon > 0$. For any $n \in \mathbb{N}$, choose a function ρ_n that is admissible for Γ_n in the definition of $\mathrm{Mod}_2(\Gamma_n)$ and such that $\int \rho_n^2\,\mathrm{d}\mathfrak{m} \leq \mathrm{Mod}_2(\Gamma_n) + \varepsilon/2^n$. Now call $\rho := \sup_n \rho_n$. Clearly ρ is admissible for $\Gamma := \bigcup_n \Gamma_n$ and it holds that

$$\mathrm{Mod}_2(\Gamma) \leq \int \rho^2\,\mathrm{d}\mathfrak{m} \leq \sum_{n\in\mathbb{N}} \int \rho_n^2\,\mathrm{d}\mathfrak{m} \leq \sum_{n\in\mathbb{N}} \mathrm{Mod}_2(\Gamma_n) + 2\,\varepsilon,$$

whence $\mathrm{Mod}_2(\Gamma) \leq \sum_{n\in\mathbb{N}} \mathrm{Mod}_2(\Gamma_n)$ by arbitrariness of ε. Hence Mod_2 is an outer measure.

Remark 11.2 Let us fix a Borel function $G : \mathrm{X} \to [0,+\infty)$ such that $G \in L^2(\mathfrak{m})$. We stress that G is everywhere defined, not an equivalence class. Then $\int_\gamma G < +\infty$ for 2-a.e. γ.

Indeed, call $\Gamma := \big\{\gamma \in \Gamma(\mathrm{X}) : \int_\gamma G = +\infty\big\}$. Given any $\varepsilon > 0$, we have that $\rho := \varepsilon\,G$ is admissible for Γ, so that $\mathrm{Mod}_2(\Gamma) \leq \varepsilon^2 \int G^2\,\mathrm{d}\mathfrak{m}$. By letting $\varepsilon \searrow 0$, we thus finally conclude that $\mathrm{Mod}_2(\Gamma) = 0$, as required. ■

Definition 11.3 (2-weak upper gradient) *Let $f : \mathrm{X} \to \mathbb{R} \cup \{\pm\infty\}$ and $G : \mathrm{X} \to [0, +\infty]$ be Borel functions, with $G \in L^2(\mathfrak{m})$. Then we say that G is a* 2-weak upper gradient *for f if*

$$\big|f(\gamma_F) - f(\gamma_I)\big| \leq \int_\gamma G \quad \text{holds for 2-a.e. } \gamma, \tag{11.4}$$

meaning that $\int_\gamma G$ must equal $+\infty$ as soon as either $\big|f(\gamma_I)\big| = +\infty$ or $\big|f(\gamma_F)\big| = +\infty$.

Remark 11.4 Consider two sets $\Gamma, \Gamma' \subseteq \Gamma(\mathrm{X})$ with the following property: for every $\gamma \in \Gamma$, there exists a subcurve of γ that belongs to Γ'. Then $\mathrm{Mod}_2(\Gamma) \leq \mathrm{Mod}_2(\Gamma')$.

The validity of such fact easily follows from the observation that any function ρ that is admissible for Γ' is admissible even for Γ. ■

Lemma 11.5 *Let G be a 2-weak upper gradient for f. Then for 2-a.e. curve $\gamma \in \Gamma(\mathrm{X})$ it holds that $\mathrm{Dom}(\gamma) \ni t \mapsto f(\gamma_t)$ is AC and $\big|\partial_t (f \circ \gamma)_t\big| \leq G(\gamma_t)|\dot{\gamma}_t|$ for a.e. $t \in \mathrm{Dom}(\gamma)$.*

Proof. Let us denote by Γ the set of curves γ for which the thesis fails. Moreover, call

$$\begin{aligned}
\Gamma' &:= \left\{\gamma \in \Gamma(\mathrm{X}) \;\middle|\; \big|f(\gamma_F) - f(\gamma_I)\big| > \int_\gamma G\right\}, \\
\widetilde{\Gamma} &:= \left\{\gamma \in \Gamma(\mathrm{X}) \;\middle|\; \int_\gamma G = +\infty\right\}.
\end{aligned}$$

Notice that $\mathrm{Mod}_2(\Gamma') = 0$ because G is a 2-weak upper gradient for f, while $\mathrm{Mod}_2(\widetilde{\Gamma}) = 0$ by Remark 11.2. Now fix $\gamma \in \Gamma \setminus \widetilde{\Gamma}$, in particular $t \mapsto G(\gamma_t)|\dot{\gamma}_t|$ belongs to $L^1(0,1)$. Then there exists $t, s \in \mathrm{Dom}(\gamma)$, $s < t$ such that $\big|f(\gamma_t) - f(\gamma_s)\big| > \int_s^t G(\gamma_r)|\dot{\gamma}_r| \,\mathrm{d}r$: if not, then γ would satisfy the thesis of the lemma. Therefore $\gamma|_{[s,t]} \in \Gamma'$, whence $\mathrm{Mod}_2(\Gamma \setminus \widetilde{\Gamma}) \leq \mathrm{Mod}_2(\Gamma')$ by Remark 11.4. This grants that $\mathrm{Mod}_2(\Gamma) \leq \mathrm{Mod}_2(\Gamma') + \mathrm{Mod}_2(\Gamma \cap \widetilde{\Gamma}) = 0$, as desired. □

We thus deduce from the previous lemma the following locality property:

Proposition 11.6 *Let G_1, G_2 be 2-weak upper gradients of f. Then $\min\{G_1, G_2\}$ is a 2-weak upper gradient of f as well.*

Proof. For $i = 1, 2$, call Γ_i the set of $\gamma \in \Gamma(\mathrm{X})$ such that $f \circ \gamma$ is AC and $\big|\partial_t (f \circ \gamma)\big| \leq G_i(\gamma_t)|\dot{\gamma}_t|$ holds for a.e. $t \in \mathrm{Dom}(\gamma)$. Then for every curve $\gamma \in \Gamma_1 \cap \Gamma_2$ we have that $f \circ \gamma$ is AC and that $\big|\partial_t (f \circ \gamma)\big| \leq \min\big\{G_1(\gamma_t), G_2(\gamma_t)\big\}|\dot{\gamma}_t|$ holds for a.e. $t \in \mathrm{Dom}(\gamma)$. By integrating such inequality over $\mathrm{Dom}(\gamma)$ we get

$$\big|f(\gamma_F) - f(\gamma_I)\big| \leq \int_\gamma \min\{G_1, G_2\} \quad \text{for every } \gamma \in \Gamma_1 \cap \Gamma_2.$$

The thesis follows by simply noticing that $\mathrm{Mod}_2\big(\Gamma(\mathrm{X}) \setminus (\Gamma_1 \cap \Gamma_2)\big) = 0$. □

Theorem 11.7 (Fuglede's lemma) *Let $G, G_n : \mathrm{X} \to [0, +\infty]$, $n \in \mathbb{N}$ be Borel functions that belong to $L^2(\mathfrak{m})$ and satisfy $\lim_n \|G_n - G\|_{L^2(\mathfrak{m})} = 0$. Then there is a subsequence $(n_k)_k$ such that $\int_\gamma |G_{n_k} - G| \overset{k}{\to} 0$ holds for 2-a.e. γ. In particular, $\int_\gamma G_{n_k} \overset{k}{\to} \int_\gamma G$ for 2-a.e. γ.*

Proof. Up to subsequence, assume that $\|G_n - G\|_{L^2(\mathfrak{m})} \le 1/2^n$ for every $n \in \mathbb{N}$. Let us define

$$\Gamma_k := \left\{ \gamma \in \Gamma(\mathrm{X}) \;\middle|\; \varlimsup_{n\to\infty} \int_\gamma |G_n - G| > \frac{1}{k} \right\} \qquad \text{for every } k \in \mathbb{N} \setminus \{0\}.$$

Observe that $\int_\gamma |G_n - G| \to 0$ as $n \to \infty$ for every $\gamma \notin \bigcup_k \Gamma_k$, thus to prove the thesis it is sufficient to show that $\mathrm{Mod}_2(\Gamma_k) = 0$ for any $k \ge 1$. Fix $k \ge 1$. For any $m \in \mathbb{N}$, let us define the function ρ_m as $\rho_m := k \sum_{n \ge m} |G_n - G|$. For every curve $\gamma \in \Gamma_k$ there is $n \ge m$ such that $\int_\gamma |G_n - G| \ge 1/k$, whence $\int_\gamma \rho_m \ge 1$, in other words ρ_m is admissible for Γ_k. Moreover, one has that $\|\rho_m\|_{L^2(\mathfrak{m})} \le k \sum_{n \ge m} \|G_n - G\|_{L^2(\mathfrak{m})} \le k/2^{m-1}$ for every $m \in \mathbb{N}$. Hence $\mathrm{Mod}_2(\Gamma_k) \le \|\rho_m\|^2_{L^2(\mathfrak{m})} \overset{m}{\to} 0$, getting the thesis. □

Theorem 11.8 *Given any $n \in \mathbb{N}$, let G_n be a 2-weak upper gradient for some function f_n. Suppose further that $G_n \to G$ and $f_n \to f$ in $L^2(\mathfrak{m})$, for suitable Borel functions $f : \mathrm{X} \to \mathbb{R}$ and $G : \mathrm{X} \to [0, +\infty]$. Then there is a Borel function $\bar{f} : \mathrm{X} \to \mathbb{R}$ such that $\bar{f}(x) = f(x)$ holds for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$ and G is a 2-weak upper gradient for $\bar{f}$.*

Proof. Possibly passing to a not relabeled subsequence, we can assume wlog that $f_n \to f$ in the $\mathfrak{m}$-a.e. sense. In addition, we can also suppose that $\int_\gamma |G_n - G| \to 0$ holds for 2-a.e. γ by Theorem 11.7. Call $\tilde{f}(x) := \varlimsup_n f_n(x)$ for every $x \in \mathrm{X}$. Then $\tilde{f} = f$ holds $\mathfrak{m}$-a.e. in X, thus accordingly $\tilde{f} \in L^2(\mathfrak{m})$. Let us define

$$\begin{aligned}
\Gamma &:= \left\{ \gamma \in \Gamma(\mathrm{X}) \;\middle|\; \int_\gamma |G_n - G| \overset{n}{\to} 0,\ f_n \circ \gamma \text{ is AC},\ \big|(f_n \circ \gamma)'\big| \le G_n \circ \gamma\, |\dot\gamma| \text{ for all } n \in \mathbb{N} \right\}, \\
\Gamma' &:= \left\{ \gamma \in \Gamma(\mathrm{X}) \;\middle|\; \text{either } \big|\tilde{f}(\gamma_I)\big| < +\infty \text{ or } \big|\tilde{f}(\gamma_F)\big| < +\infty \right\}, \\
\widetilde{\Gamma} &:= \left\{ \gamma \in \Gamma(\mathrm{X}) \;\middle|\; \big|\tilde{f}(\gamma_t)\big| = +\infty \text{ for every } t \in \mathrm{Dom}(\gamma) \right\}.
\end{aligned}$$

Note that $\mathrm{Mod}_2(\Gamma^c) = 0$ because G_n is a 2-weak upper gradient of f_n for any $n \in \mathbb{N}$. Furthermore, we have that $\mathrm{Mod}_2(\widetilde{\Gamma}) = 0$: indeed, for every $\varepsilon > 0$ the function $\rho := \varepsilon|\tilde{f}|$ is admissible for $\widetilde{\Gamma}$ and $\|\rho\|_{L^2(\mathfrak{m})} \le \varepsilon \|f\|_{L^2(\mathfrak{m})}$. We now claim that

$$\big|\tilde{f}(\gamma_F) - \tilde{f}(\gamma_I)\big| \le \int_\gamma G \qquad \text{for every } \gamma \in \Gamma \cap \Gamma'. \tag{11.5}$$

To prove it, just observe that $\big|\tilde{f}(\gamma_F) - \tilde{f}(\gamma_I)\big| \le \varlimsup_n \big|f_n(\gamma_F) - f_n(\gamma_I)\big| \le \lim_n \int_\gamma G_n = \int_\gamma G$ for every $\gamma \in \Gamma \cap \Gamma'$. We can use (11.5) to prove that

$$\big|\tilde{f}(\gamma_F) - \tilde{f}(\gamma_I)\big| \le \int_\gamma G \qquad \text{for every } \gamma \in \Gamma \setminus \widetilde{\Gamma}. \tag{11.6}$$

Indeed: fix $\gamma \in \Gamma \setminus \widetilde{\Gamma}$. There exists $t_0 \in \mathrm{Dom}(\gamma)$ such that $\big|\tilde{f}(\gamma_{t_0})\big| < +\infty$. Call $\gamma^1 := \gamma|_{[I,t_0]}$ and $\gamma^2 := \gamma|_{[t_0,F]}$. We have that $\gamma^1, \gamma^2 \in \Gamma \cap \Gamma'$, so that (11.5) yields

$$\big|\tilde{f}(\gamma_F) - \tilde{f}(\gamma_I)\big| \le \big|\tilde{f}(\gamma_F) - \tilde{f}(\gamma_{t_0})\big| + \big|\tilde{f}(\gamma_{t_0}) - \tilde{f}(\gamma_{t_0})\big| \le \int_{\gamma^1} G + \int_{\gamma^2} G = \int_\gamma G.$$

Since $\mathrm{Mod}_2\big(\Gamma(\mathrm{X}) \setminus (\Gamma \setminus \widetilde{\Gamma})\big) = 0$, we deduce from (11.6) that G is a 2-weak upper gradient of the function $\bar{f} : \mathrm{X} \to \mathbb{R}$, defined by $\bar{f} := \chi_{\{\tilde{f} < +\infty\}} \tilde{f}$, which $\mathfrak{m}$-a.e. coincides with f. □

We now define the Sobolev space $W^{1,2}_{\mathrm{Sh}}(\mathrm{X})$, where 'Sh' stays for Shanmugalingam, who first introduced such object.

Definition 11.9 *We define the Sobolev space $W^{1,2}_{\mathrm{Sh}}(\mathrm{X})$ as the set of all $f \in L^2(\mathfrak{m})$ such that there exist two Borel functions $\bar{f} : \mathrm{X} \to \mathbb{R}$ and $G : \mathrm{X} \to [0,+\infty]$ in $L^2(\mathfrak{m})$ satisfying these properties: $\bar{f}(x) = f(x)$ for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$ and G is a 2-weak upper gradient for $\bar{f}$.*

We endow the vector space $W^{1,2}_{\mathrm{Sh}}(\mathrm{X})$ with the norm given by

$$\|f\|^2_{W^{1,2}_{\mathrm{Sh}}(\mathrm{X})} := \|f\|^2_{L^2(\mathfrak{m})} + \inf \|G\|^2_{L^2(\mathfrak{m})} \quad \textit{for every } f \in W^{1,2}_{\mathrm{Sh}}(\mathrm{X}), \tag{11.7}$$

where the infimum is taken among all Borel functions $G : \mathrm{X} \to [0,+\infty]$ that are 2-weak upper gradients of some Borel representative of f.

Remark 11.10 (Minimal 2-weak upper gradient) Given any $f \in W^{1,2}_{\mathrm{Sh}}(\mathrm{X})$, there exists a *minimal 2-weak upper gradient* $|Df|_{\mathrm{Sh}}$, where minimality has to be intended in the $\mathfrak{m}$-a.e. sense. In other words, if $\bar{f}$ is a Borel representative of f and G is a 2-weak upper gradient for $\bar{f}$, then $|Df|_{\mathrm{Sh}} \leq G$ holds $\mathfrak{m}$-a.e. in X. It thus holds that

$$\|f\|^2_{W^{1,2}_{\mathrm{Sh}}(\mathrm{X})} = \|f\|^2_{L^2(\mathfrak{m})} + \big\||Df|_{\mathrm{Sh}}\big\|^2_{L^2(\mathfrak{m})} \quad \text{for every } f \in W^{1,2}_{\mathrm{Sh}}(\mathrm{X}). \tag{11.8}$$

These statements follow from Proposition 11.6 and Theorem 11.8. ■

Lemma 11.11 *Let Γ be a subset of $AC([0,1],\mathrm{X})$ such that $\mathrm{Mod}_2(\Gamma) = 0$. Then $\boldsymbol{\pi}^*(\Gamma) = 0$ for every test plan $\boldsymbol{\pi}$ on X, where $\boldsymbol{\pi}^*$ denotes the outer measure induced by $\boldsymbol{\pi}$.*

Proof. Take ρ admissible for Γ. The function $(\gamma,t) \mapsto \rho(\gamma_t)|\dot{\gamma}_t|$ is Borel, hence $\big\{\gamma : \int_\gamma \rho \geq 1\big\}$ is a $\boldsymbol{\pi}$-measurable set by Fubini theorem. Observe that such set contains Γ, so that

$$\begin{aligned}
\boldsymbol{\pi}^*(\Gamma) &\leq \iint_\gamma \rho \, \mathrm{d}\boldsymbol{\pi}(\gamma) = \int_0^1\!\!\int \rho(\gamma_t)|\dot{\gamma}_t| \, \mathrm{d}\boldsymbol{\pi}(\gamma)\, \mathrm{d}t \\
&\leq \left(\int_0^1\!\!\int \rho^2(\gamma_t) \, \mathrm{d}\boldsymbol{\pi}(\gamma)\, \mathrm{d}t\right)^{1/2} \left(\int_0^1\!\!\int |\dot{\gamma}_t|^2 \, \mathrm{d}\boldsymbol{\pi}(\gamma)\, \mathrm{d}t\right)^{1/2} \\
&\leq \sqrt{\mathrm{Comp}(\boldsymbol{\pi})} \left(\int_0^1\!\!\int |\dot{\gamma}_t|^2 \, \mathrm{d}\boldsymbol{\pi}(\gamma)\, \mathrm{d}t\right)^{1/2} \left(\int \rho^2 \, \mathrm{d}\mathfrak{m}\right)^{1/2}.
\end{aligned}$$

By arbitrariness of ρ, we conclude that $\boldsymbol{\pi}^*(\Gamma) = 0$. □

Remark 11.12 It holds that

$$\begin{aligned}
&|Df|_{*,a} \geq |Df|_* \geq |Df|_{\mathrm{Ch}} \geq |Df|_{\mathrm{Sh}} \geq |Df|, \\
&W^{1,2}_{*,a}(\mathrm{X}) \subseteq W^{1,2}_*(\mathrm{X}) \subseteq W^{1,2}_{\mathrm{Ch}}(\mathrm{X}) \subseteq W^{1,2}_{\mathrm{Sh}}(\mathrm{X}) \subseteq W^{1,2}(\mathrm{X}).
\end{aligned} \tag{11.9}$$

To prove $|Df|_{\mathrm{Ch}} \geq |Df|_{\mathrm{Sh}}$, observe that any upper gradient is a 2-weak upper gradient. On the other hand, to show $|Df|_{\mathrm{Sh}} \geq |Df|$ it suffices to apply Lemma 11.11. ■

Theorem 11.13 (Ambrosio-Gigli-Savaré) *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a metric measure space. Then Lipschitz functions in* X *are* dense in energy *in* $W^{1,2}(\mathrm{X})$*, namely for every* $f\in W^{1,2}(\mathrm{X})$ *there exists a sequence* $(f_n)_n\subseteq \mathrm{LIP}(\mathrm{X})\cap L^2(\mathfrak{m})$ *such that* $f_n\to f$ *and* $\mathrm{lip}_a(f_n)\to|Df|$ *in* $L^2(\mathfrak{m})$*, thus accordingly also* $\mathrm{lip}(f_n)\to|Df|$ *and* $|Df_n|\to|Df|$ *in* $L^2(\mathfrak{m})$*.*

In particular, we have that $W^{1,2}_{*,a}(\mathrm{X})=W^{1,2}(\mathrm{X})$ *and that the equality* $|Df|_{*,a}=|Df|$ *is satisfied* $\mathfrak{m}$*-a.e. for every* $f\in W^{1,2}(\mathrm{X})$*.*

We directly deduce from Theorem 11.13 that all inequalities and inclusions in (11.9) are actually equalities. In other words, all the several approaches we saw are equivalent.

Remark 11.14 In order to prove that $|Df|_{\mathrm{Ch}}=|Df|_{\mathrm{Sh}}$, the following fact is sufficient:

$$\text{Let } G \text{ be a 2-weak upper gradient for } f \text{ and let } \varepsilon>0. \text{ Then there exists an upper gradient } \widetilde{G} \text{ for } f \text{ such that } \|\widetilde{G}\|_{L^2(\mathfrak{m})}\le\|G\|_{L^2(\mathfrak{m})}+\varepsilon. \tag{11.10}$$

To prove it: call Γ the set of $\gamma\in\Gamma(\mathrm{X})$ such that $\big|f(\gamma_F)-f(\gamma_I)\big|>\int_\gamma G$, so that $\mathrm{Mod}_2(\Gamma)=0$. We first need to show that

$$\exists\,\rho:\ \mathrm{X}\to[0,+\infty] \text{ Borel such that } \int_\gamma\rho=+\infty \text{ for all } \gamma\in\Gamma \text{ and } \|\rho\|_{L^2(\mathfrak{m})}\le\varepsilon. \tag{11.11}$$

There exists $(\rho_n)_n$ such that $\int_\gamma\rho_n\ge 1$ and $\|\rho_n\|_{L^2(\mathfrak{m})}\le\varepsilon/2^n$ for all $n\in\mathbb{N}$ and $\gamma\in\Gamma$. Thus it can be easily seen that the function $\rho:=\sum_{n\ge1}\rho_n$ satisfies (11.11): for every $\gamma\in\Gamma$ we have that $\int_\gamma\rho=\lim_{m\to\infty}\sum_{n=1}^m\int_\gamma\rho_n\ge\lim_{m\to\infty}m=+\infty$, while $\|\rho\|_{L^2(\mathfrak{m})}\le\sum_{n\ge1}\|\rho_n\|_{L^2(\mathfrak{m})}\le\varepsilon$.

Finally, let us call $\widetilde{G}:=G+\rho$. Clearly $\widetilde{G}$ satisfies (11.10): if $\gamma\in\Gamma$ then $\int_\gamma\widetilde{G}=+\infty$, while if $\gamma\notin\Gamma$ then $\big|f(\gamma_F)-f(\gamma_I)\big|\le\int_\gamma G\le\int_\gamma\widetilde{G}$, i.e. $\widetilde{G}$ is an upper gradient of f; moreover, one has $\|\widetilde{G}\|_{L^2(\mathfrak{m})}\le\|G\|_{L^2(\mathfrak{m})}+\|\rho\|_{L^2(\mathfrak{m})}\le\|G\|_{L^2(\mathfrak{m})}+\varepsilon$. This concludes the proof. ■

12 Lesson [11/12/2017]

Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a fixed metric measure space.

Definition 12.1 (L^2-normed L^∞-module) *We define an* $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module*, or briefly* module*, as a quadruplet* $\big(\mathscr{M},\|\cdot\|_{\mathscr{M}},\cdot,|\cdot|\big)$ *with the following properties:*

i) $\big(\mathscr{M},\|\cdot\|_{\mathscr{M}}\big)$ *is a Banach space,*

ii) *the* multiplication by L^∞-functions $\cdot:\ L^\infty(\mathfrak{m})\times\mathscr{M}\to\mathscr{M}$ *is a bilinear map satisfying*

$$\begin{aligned} f\cdot(g\cdot v)&=(fg)\cdot v \quad \text{for every } f,g\in L^\infty(\mathfrak{m}) \text{ and } v\in\mathscr{M},\\ \hat{1}\cdot v&=v \quad \text{for every } v\in\mathscr{M}, \end{aligned} \tag{12.1}$$

where $\hat{1}$ *denotes the (equivalence class of the) function on* X *identically equal to* 1*,*

iii) *the* pointwise norm $|\cdot|:\mathscr{M}\to L^2(\mathfrak{m})$ *satisfies*

$$\begin{aligned} |v| &\geq 0 \quad \mathfrak{m}\text{-}a.e. && \textit{for every } v\in\mathscr{M}, \\ |f\cdot v| &= |f||v| \quad \mathfrak{m}\text{-}a.e. && \textit{for every } f\in L^\infty(\mathfrak{m}) \textit{ and } v\in\mathscr{M}, \\ \|v\|_{\mathscr{M}} &= \big\||v|\big\|_{L^2(\mathfrak{m})} && \textit{for every } v\in\mathscr{M}. \end{aligned} \tag{12.2}$$

For the sake of brevity, we shall often write fv instead of $f\cdot v$.

Proposition 12.2 *Let $\mathscr{M}$ be a module. Then:*

i) $\|fv\|_{\mathscr{M}}\leq\|f\|_{L^\infty(\mathfrak{m})}\|v\|_{\mathscr{M}}$ *for every $f\in L^\infty(\mathfrak{m})$ and $v\in\mathscr{M}$.*

ii) $\lambda v=\hat{\lambda}v$ *for every $\lambda\in\mathbb{R}$, where $\hat{\lambda}$ denotes the (equivalence class of the) function on* X *identically equal to λ.*

iii) *It holds that*

$$\begin{aligned} |v+w| &\leq |v|+|w| \\ |\lambda v| &= |\lambda||v| \end{aligned} \quad \mathfrak{m}\text{-}a.e. \quad \textit{for every } v,w\in\mathscr{M} \textit{ and } \lambda\in\mathbb{R}. \tag{12.3}$$

Proof. i) Simply notice that

$$\|fv\|_{\mathscr{M}}=\big\||f||v|\big\|_{L^2(\mathfrak{m})}\leq\|f\|_{L^\infty(\mathfrak{m})}\big\||v|\big\|_{L^2(\mathfrak{m})}=\|f\|_{L^\infty(\mathfrak{m})}\|v\|_{\mathscr{M}}$$

is verified for every $f\in L^\infty(\mathfrak{m})$ and $v\in\mathscr{M}$ by (12.2) and by Hölder inequality.
ii) Given any $\lambda\in\mathbb{R}$ and $v\in\mathscr{M}$, we have that $\hat{\lambda}v=(\lambda\hat{1})v=\lambda(\hat{1}v)=\lambda v$ by (12.1) and by bilinearity of the multiplication by L^∞-functions.
iii) Fix $\lambda\in\mathbb{R}$ and $v,w\in\mathscr{M}$. Clearly $|\lambda v|=|\hat{\lambda}v|=|\hat{\lambda}||v|=|\lambda||v|$ holds $\mathfrak{m}$-a.e. in X as a consequence of ii). On the other hand, in order to prove that $|v+w|\leq|v|+|w|$ holds $\mathfrak{m}$-a.e. we argue by contradiction: suppose the contrary, thus there exist $a,b,c\in\mathbb{R}$ with $a+b<c$ and $E\subseteq$ X Borel with $\mathfrak{m}(E)>0$ such that

$$\begin{cases} |v|\leq a \\ |w|\leq b \\ |v+w|\geq c \end{cases} \qquad \text{holds } \mathfrak{m}\text{-a.e. in } E. \tag{12.4}$$

Hence we deduce from (12.4) that

$$\begin{aligned} \big\|\chi_E(v+w)\big\|_{\mathscr{M}} &= \left(\int_E|v+w|^2\,\mathrm{d}\mathfrak{m}\right)^{1/2}\geq c\,\mathfrak{m}(E)^{1/2}>(a+b)\,\mathfrak{m}(E)^{1/2} \\ &\geq\left(\int_E|v|^2\,\mathrm{d}\mathfrak{m}\right)^{1/2}+\left(\int_E|w|^2\,\mathrm{d}\mathfrak{m}\right)^{1/2}=\|\chi_E\,v\|_{\mathscr{M}}+\|\chi_E\,w\|_{\mathscr{M}}, \end{aligned}$$

which contradicts the fact the $\|\cdot\|_{\mathscr{M}}$ is a norm. Therefore (12.3) is proved. $\square$

Exercise 12.3 Let V, W, Z be normed spaces. Let $B : V \times W \to X$ be a bilinear operator.

i) Show that B is continuous if and only if both $B(v, \cdot)$ and $B(\cdot, w)$ are continuous for every $v \in V$ and $w \in W$.

ii) Prove that B is continuous if and only if there exists a constant $C > 0$ such that the inequality $\|B(v,w)\|_Z \leq C\,\|v\|_V \|w\|_W$ holds for every $(v,w) \in V \times W$.

Remark 12.4 It directly follows from property i) of Proposition 12.2 and from Exercise 12.3 that the multiplication by L^∞-functions is a continuous operator. ■

Example 12.5 We provide some examples of $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-modules:

i) The space $L^2(\mathfrak{m})$ itself can be viewed as a module.

ii) More in general, the space $L^2(\mathrm{X}, \mathbb{B})$ is a module for every Banach space $\mathbb{B}$. (In the case in which $\mathfrak{m}$ is a finite measure, the space $L^2(\mathrm{X}, \mathbb{B})$ is defined as the set of all elements v of $L^1(\mathrm{X}, \mathbb{B})$ for which the quantity $\int \|v(x)\|^2_{\mathbb{B}}\,\mathrm{d}\mathfrak{m}(x)$ is finite.)

iii) The space of L^2-vector fields on a Riemannian manifold is a module with respect to the pointwise operations. Actually, the same holds true even for a Finsler manifold (i.e., roughly speaking, a manifold endowed with a norm on each tangent space).

iv) The space of L^2-sections of a 'measurable bundle' over X (whose fibers are Banach spaces) has a natural structure of L^2-normed L^∞-module. For instance, consider the spaces of covector fields or higher dimensional tensors with pointwise norm in $L^2(\mathfrak{m})$.

Remark 12.6 One can imagine a module $\mathscr{M}$, in a sense, as the space of L^2-sections of some measurable Banach bundle over X. Cf. Serre-Swan theorem. ■

Definition 12.7 *Let $\mathscr{M}$ be a module and $v \in \mathscr{M}$. Then let us define*

$$\{v = 0\} := \big\{|v| = 0\big\}. \tag{12.5}$$

Notice that $\{v = 0\}$ is a Borel set in X, *defined up to $\mathfrak{m}$-a.e. equality. Similarly, one can define $\{v \neq 0\}$, $\{v = w\}$ for $w \in \mathscr{M}$ and so on.*

It is trivial to check that for any $E \subseteq \mathrm{X}$ Borel one has

$$\chi_E\, v = 0 \quad \Longleftrightarrow \quad |v| = 0 \quad \mathfrak{m}\text{-a.e. in } E. \tag{12.6}$$

Indeed, $\chi_E\, v = 0$ iff $\|\chi_E\, v\|_{\mathscr{M}} = 0$ iff $\int_E |v|^2\,\mathrm{d}\mathfrak{m} = 0$ iff $|v| = 0$ holds $\mathfrak{m}$-a.e. in E.

If (one of) the two conditions in (12.6) hold, we say that v *is $\mathfrak{m}$-a.e. null in E*.

Remark 12.8 Let $\mathscr{M}$ be a module. Let $v \in \mathscr{M}$. Suppose to have a sequence $(E_n)_n$ of Borel subsets of X such that $\chi_{E_n} v = 0$ for every $n \in \mathbb{N}$. Then v is $\mathfrak{m}$-a.e. null in $\bigcup_n E_n$, as one can readily deduce from the characterisation (12.6). ■

Proposition 12.9 ($\mathfrak{m}$-essential union) *Let $\{E_i\}_{i\in I}$ be a (not necessarily countable) family of Borel subsets of* X*. Then there exists a Borel set $E \subseteq \mathrm{X}$ such that*

i) $\mathfrak{m}(E_i \setminus E) = 0$ *for every $i \in I$,*

ii) *if $F \subseteq \mathrm{X}$ Borel satisfies $\mathfrak{m}(E_i \setminus F) = 0$ for all $i \in I$, then $\mathfrak{m}(E \setminus F) = 0$.*

Such set E, which is called the $\mathfrak{m}$-essential union *of $\{E_i\}_{i\in I}$, is $\mathfrak{m}$-a.e. unique, in the sense that any other Borel set $\widetilde{E}$ with the same properties must satisfy $\mathfrak{m}(E\Delta\widetilde{E}) = 0$.*

Proof. Uniqueness follows from condition ii). To prove existence, assume wlog that $\mathfrak{m} \in \mathscr{P}(\mathrm{X})$ (otherwise, replace $\mathfrak{m}$ with a Borel probability measure $\widetilde{\mathfrak{m}}$ such that $\widetilde{\mathfrak{m}} \ll \mathfrak{m} \ll \widetilde{\mathfrak{m}}$, which can be built as in the proof of STEP 5 of Theorem 8.7). Denote by $\mathcal{A}$ the family of all finite unions of the E_i's and call $S := \sup\big\{\mathfrak{m}(A) \,:\, A \in \mathcal{A}\big\}$. Therefore there exists an increasing sequence of sets $(A_n)_n \subseteq \mathcal{A}$ such that $\mathfrak{m}(A_n) \nearrow S$. Let us now define $E := \bigcup_n A_n$. Clearly E satisfies i): if not, there exists some $i \in I$ such that $\mathfrak{m}(E_i \setminus E) > 0$, whence

$$S = \mathfrak{m}(E) < \mathfrak{m}(E \cup E_i) = \lim_{n\to\infty} \mathfrak{m}(A_n \cup E_i) \leq S,$$

which leads to a contradiction. Moreover, the set E can be clearly written as countable union of elements in $\{E_i\}_{i\in I}$, say $E = \bigcup_{j\in J} E_j$ for some $J \subseteq I$ countable. Hence for any $F \subseteq \mathrm{X}$ Borel with $\mathfrak{m}(E_i \setminus F) = 0$ for each $i \in I$, it holds that

$$\mathfrak{m}(E \setminus F) \leq \sum_{j\in J} \mathfrak{m}(E_j \setminus F) = 0,$$

proving ii) and accordingly the existence part of the statement. □

Given any $v \in \mathscr{M}$, it holds that $\{v = 0\}$ can equivalently described as the $\mathfrak{m}$-essential union of all Borel sets $E \subseteq \mathrm{X}$ such that $\chi_E\, v = 0$.

Example 12.10 Define $E_i := \{i\}$ for every $i \in \mathbb{R}$. Then the set-theoretic union of $\{E_i\}_{i\in\mathbb{R}}$ is the whole real line $\mathbb{R}$, while its $\mathcal{L}^1$-essential union is given by the empty set.

Definition 12.11 (Localisation of a module) *Let $\mathscr{M}$ be a module. Let E be any Borel subset of* X*. Then we define*

$$\mathscr{M}|_E := \big\{\chi_E\, v \,:\, v \in \mathscr{M}\big\} \subseteq \mathscr{M}. \tag{12.7}$$

It turns out that $\mathscr{M}|_E$ is stable under all module operations and is complete, thus it is a submodule of $\mathscr{M}$.

Proposition 12.12 *Let S be any subset of $\mathscr{M}$. Let us define*

$$\mathscr{M}(S) := \mathscr{M}\text{-closure of } \mathcal{S} := \left\{ \sum_{i=1}^n f_i\, v_i \;\middle|\; n \in \mathbb{N},\, (f_i)_{i=1}^n \subseteq L^\infty(\mathfrak{m}),\, (v_i)_{i=1}^n \subseteq S \right\}. \tag{12.8}$$

Then $\mathscr{M}(S)$ is the smallest submodule of $\mathscr{M}$ containing S.

Proof. We omit the simple proof of the fact that $\mathscr{M}(S)$ inherits from $\mathscr{M}$ a module structure. Moreover, any module containing the set S must contain also $\mathcal{S}$ and must be closed, whence the required minimality. □

Definition 12.13 (Generators) *The module $\mathscr{M}(S)$ that we defined in Proposition 12.12 is called the* module generated by S. *Moreover, if $E \subseteq \mathrm{X}$ is Borel and $\mathscr{M}(S)_{|E} = \mathscr{M}_{|E}$, then we say that S* generates $\mathscr{M}$ on E.

Remark 12.14 The space $L^2(\mathfrak{m})$, viewed as a module, can be generated by a single element, namely by any $L^2(\mathfrak{m})$-function which is $\mathfrak{m}$-a.e. different from 0. ■

Proposition 12.15 *Let V be a vector subspace of $\mathscr{M}$. Then $\mathscr{M}(V)$ is the $\mathscr{M}$-closure of*

$$\mathcal{V} := \left\{ \sum_{i=1}^{n} \chi_{E_i} v_i \;\middle|\; n \in \mathbb{N},\ (E_i)_{i=1}^{n} \text{ Borel partition of } \mathrm{X},\ (v_i)_{i=1}^{n} \subseteq V \right\}. \tag{12.9}$$

Proof. The inclusion $\mathrm{cl}_{\mathscr{M}}(\mathcal{V}) \subseteq \mathscr{M}(V)$ is trivial. To prove the converse inclusion, since $\mathcal{V}$ and accordingly also $\mathrm{cl}_{\mathscr{M}}(\mathcal{V})$ are vector spaces, it suffices to show that $f\,v \in \mathrm{cl}_{\mathscr{M}}(\mathcal{V})$ whenever we have $f \in L^\infty(\mathfrak{m})$ and $v \in V \setminus \{0\}$. Given any $\varepsilon > 0$, pick a simple function $g = \sum_{i=1}^n \alpha_i\,\chi_{E_i}$ such that $\|f - g\|_{L^\infty(\mathfrak{m})} \leq \varepsilon/\|v\|_{\mathscr{M}}$. Then $\|f\,v - g\,v\|_{\mathscr{M}} \leq \varepsilon$ and $g\,v = \sum_{i=1}^n \chi_{E_i}(\alpha_i\,v) \in \mathcal{V}$, as required. Hence the thesis is achieved. □

13 Lesson [13/12/2017]

Remark 13.1 Let $\mathscr{M}$ be a module. Then the pointwise norm $|\cdot| : \mathscr{M} \to L^2(\mathfrak{m})$ is continuous. Indeed, since $\big||v| - |w|\big| \leq |v - w|$ holds $\mathfrak{m}$-a.e. for any $v, w \in \mathscr{M}$ by (12.3), one immediately deduces that $\big\||v| - |w|\big\|_{L^2(\mathfrak{m})} \leq \|v - w\|_{\mathscr{M}}$ for every $v, w \in \mathscr{M}$. ■

Theorem 13.2 (Cotangent module) *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a fixed metric measure space. Then there exists a unique couple $\big(L^2(T^*\mathrm{X}), \mathrm{d}\big)$, where $L^2(T^*\mathrm{X})$ is an $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module and $\mathrm{d} : \mathrm{S}^2(\mathrm{X}) \to L^2(T^*\mathrm{X})$ is a linear operator, such that*

i) *$|\mathrm{d}f| = |Df|$ holds $\mathfrak{m}$-a.e. for every $f \in \mathrm{S}^2(\mathrm{X})$,*

ii) *$L^2(T^*\mathrm{X})$ is generated by $\big\{\mathrm{d}f \,:\, f \in \mathrm{S}^2(\mathrm{X})\big\}$.*

Uniqueness is intended up to unique isomorphism: if another couple $(\widetilde{\mathscr{M}}, \widetilde{\mathrm{d}})$ satisfies the same properties, then there is a unique module isomorphism $\Phi : L^2(T^*\mathrm{X}) \to \widetilde{\mathscr{M}}$ such that $\Phi \circ \mathrm{d} = \widetilde{\mathrm{d}}$.

We call $L^2(T^*\mathrm{X})$ the cotangent module *associated to $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ and d the* differential.

Proof. UNIQUENESS. Fix any couple $(\widetilde{\mathscr{M}}, \widetilde{\mathrm{d}})$ that satisfies both i) and ii). We claim that for every $f, g \in \mathrm{S}^2(\mathrm{X})$ and $E \subseteq \mathrm{X}$ Borel it holds that

$$\mathrm{d}f = \mathrm{d}g \quad \mathfrak{m}\text{-a.e. on } E \quad\iff\quad \widetilde{\mathrm{d}}f = \widetilde{\mathrm{d}}g \quad \mathfrak{m}\text{-a.e. on } E. \tag{13.1}$$

Indeed, $\mathrm{d}f = \mathrm{d}g$ $\mathfrak{m}$-a.e. on E if and only if $|\mathrm{d}(f-g)| = |D(f-g)| = |\widetilde{\mathrm{d}}(f-g)|$ $\mathfrak{m}$-a.e. on E if and only if $\widetilde{\mathrm{d}}f = \widetilde{\mathrm{d}}g$ $\mathfrak{m}$-a.e. on E. Now let us define

$$V := \left\{ \sum_{i=1}^{n} \chi_{E_i} \mathrm{d}f_i \;\middle|\; n \in \mathbb{N},\ (E_i)_{i=1}^{n} \text{ Borel partition of X},\ (f_i)_{i=1}^{n} \subseteq \mathrm{S}^2(\mathrm{X}) \right\},$$

$$\widetilde{V} := \left\{ \sum_{i=1}^{n} \chi_{E_i} \widetilde{\mathrm{d}}f_i \;\middle|\; n \in \mathbb{N},\ (E_i)_{i=1}^{n} \text{ Borel partition of X},\ (f_i)_{i=1}^{n} \subseteq \mathrm{S}^2(\mathrm{X}) \right\},$$

which are vector subspaces of $L^2(T^*\mathrm{X})$ and $\widetilde{\mathscr{M}}$, respectively. Note that any module isomorphism $\Phi : L^2(T^*\mathrm{X}) \to \widetilde{\mathscr{M}}$ satisfying $\Phi \circ \mathrm{d} = \widetilde{\mathrm{d}}$ must necessarily restrict to the map $\Phi : V \to \widetilde{V}$ given by

$$\Phi\left(\sum_{i=1}^{n} \chi_{E_i} \mathrm{d} f_i\right) := \sum_{i=1}^{n} \chi_{E_i} \widetilde{\mathrm{d}} f_i \quad \text{for every } \sum_{i=1}^{n} \chi_{E_i} \mathrm{d} f_i \in V. \tag{13.2}$$

Well-posedness of (13.2) stems from (13.1). Moreover, the $\mathfrak{m}$-a.e. equalities

$$\left|\sum_{i=1}^{n} \chi_{E_i} \widetilde{\mathrm{d}} f_i\right| = \sum_{i=1}^{n} \chi_{E_i} |\widetilde{\mathrm{d}} f_i| = \sum_{i=1}^{n} \chi_{E_i} |D f_i| = \sum_{i=1}^{n} \chi_{E_i} |\mathrm{d} f_i| = \left|\sum_{i=1}^{n} \chi_{E_i} \mathrm{d} f_i\right|$$

grant that Φ preserves the pointwise norm, whence also the norm. Since V is dense in $L^2(T^*\mathrm{X})$ by property ii) for $\big(L^2(T^*\mathrm{X}), \mathrm{d}\big)$, the linear continuous map $\Phi : V \to \widetilde{\mathscr{M}}$ can be uniquely extended to an operator $\Phi : L^2(T^*\mathrm{X}) \to \widetilde{\mathscr{M}}$, which is linear continuous and preserves the pointwise norm by Remark 13.1. In particular, it is an isometry, whence it is injective and it has closed image. Given that $\Phi(V) = \widetilde{V}$ is dense in $\widetilde{\mathscr{M}}$ by property ii) for $(\widetilde{\mathscr{M}}, \widetilde{\mathrm{d}})$, we deduce that Φ is also surjective. In order to conclude, it only remains to show that Φ is $L^\infty(\mathfrak{m})$-linear. To do so, first notice that $\Phi(\chi_E\, v) = \chi_E\, \Phi(v)$ is satisfied for every $E \subseteq \mathrm{X}$ Borel and $v \in V$. Since Φ and the multiplication by L^∞-functions are continuous, the same property holds for every $v \in L^2(T^*\mathrm{X})$, whence $\Phi(f\, v) = f\, \Phi(v)$ for all $f : \mathrm{X} \to \mathbb{R}$ simple and $v \in L^2(T^*\mathrm{X})$ by linearity of Φ. Finally, the same is true also for every $f \in L^\infty(\mathfrak{m})$ by density of the simple functions in $L^\infty(\mathfrak{m})$. This completes the proof of the uniqueness part of the statement.

Existence. Let us define the *pre-cotangent module* as the set

$$\mathsf{Pcm} := \left\{ \big\{(E_i, f_i)\big\}_{i=1}^{n} \;\middle|\; n \in \mathbb{N},\ (E_i)_{i=1}^{n} \text{ Borel partition of X},\ (f_i)_{i=1}^{n} \subseteq \mathrm{S}^2(\mathrm{X}) \right\}.$$

For simplicity, we shall write $(E_i, f_i)_i$ instead of $\big\{(E_i, f_i)\big\}_{i=1}^{n}$. We introduce an equivalence relation on Pcm: we say $(E_i, f_i)_i \sim (F_j, g_j)_j$ if and only if $|D(f_i - g_j)| = 0$ $\mathfrak{m}$-a.e. in $E_i \cap F_j$ for every i, j. Let us denote by $[E_i, f_i]_i \in \mathsf{Pcm}/\sim$ the equivalence class of $(E_i, f_i)_i \in \mathsf{Pcm}$.

We now define some operations on the quotient $\mathsf{Pcm}/\sim$, which are well-defined by locality of minimal weak upper gradients (recall Theorem 8.7):

$$\begin{aligned}
[E_i, f_i]_i + [F_j, g_j]_j &:= [E_i \cap F_j, f_i + g_j]_{i,j}, \\
\alpha\, [E_i, f_i]_i &:= [E_i, \alpha\, f_i]_i, \\
\left(\sum_j \alpha_j\, \chi_{F_j}\right) \cdot [E_i, f_i]_i &:= [E_i \cap F_j, \alpha_j\, f_i]_{i,j}, \\
\big|[E_i, f_i]_i\big| &:= \sum_i \chi_{E_i} |D f_i| \quad \mathfrak{m}\text{-a.e. in X}, \\
\big\|[E_i, f_i]_i\big\| &:= \big\|\big|[E_i, f_i]_i\big|\big\|_{L^2(\mathfrak{m})} = \left(\sum_i \int_{E_i} |D f_i|^2 \,\mathrm{d}\mathfrak{m}\right)^{1/2}.
\end{aligned} \tag{13.3}$$

The first two operations in (13.3) give $\mathsf{Pcm}/\sim$ a vector space structure, the third one is the multiplication by simple functions $\cdot : \mathsf{Sf}(\mathfrak{m}) \times (\mathsf{Pcm}/\sim) \to (\mathsf{Pcm}/\sim)$ (where $\mathsf{Sf}(\mathfrak{m})$ denotes the space of all simple functions on X modulo $\mathfrak{m}$-a.e. equality), the fourth one is the pointwise norm $|\cdot|:(\mathsf{Pcm}/\sim)\to L^2(\mathfrak{m})$ and the fifth one is a norm on $\mathsf{Pcm}/\sim$.

We only prove that $\|\cdot\|$ is actually a norm on $\mathsf{Pcm}/\sim$: if $\big\|[E_i,f_i]_i\big\|=0$ then $|Df_i|=0$ holds $\mathfrak{m}$-a.e. on E_i for every i, so that $(E_i,f_i)_i\sim(\mathrm{X},0)$. Moreover, it directly follows from the definitions in (13.3) that $\big\|\alpha\,[E_i,f_i]_i\big\|=|\alpha|\,\big\|[E_i,f_i]_i\big\|$. Finally, one has

$$\begin{aligned}
\big\|[E_i,f_i]_i+[F_j,g_j]_j\big\| &= \big\|[E_i\cap F_j,f_i+g_j]_{i,j}\big\| = \Big\|\sum_{i,j}\chi_{E_i\cap F_j}\,\big|D(f_i+g_j)\big|\Big\|_{L^2(\mathfrak{m})}\\
&\leq \Big\|\sum_{i,j}\chi_{E_i\cap F_j}\,|Df_i|\Big\|_{L^2(\mathfrak{m})}+\Big\|\sum_{i,j}\chi_{E_i\cap F_j}\,|Dg_j|\Big\|_{L^2(\mathfrak{m})}\\
&= \Big\|\sum_{i}\chi_{E_i}\,|Df_i|\Big\|_{L^2(\mathfrak{m})}+\Big\|\sum_{j}\chi_{F_j}\,|Dg_j|\Big\|_{L^2(\mathfrak{m})}\\
&= \big\|[E_i,f_i]_i\big\|+\big\|[F_j,g_j]_j\big\|,
\end{aligned}$$

which is the triangle inequality for $\|\cdot\|$. Hence $\|\cdot\|$ is a norm on $\mathsf{Pcm}/\sim$.

Let us denote by $\big(L^2(T^*\mathrm{X}),\|\cdot\|_{L^2(T^*\mathrm{X})}\big)$ the completion of $\big(\mathsf{Pcm}/\sim,\|\cdot\|\big)$. One has that the operations $|\cdot|:(\mathsf{Pcm}/\sim)\to L^2(\mathfrak{m})$ and $\cdot:\mathsf{Sf}(\mathfrak{m})\times(\mathsf{Pcm}/\sim)\to(\mathsf{Pcm}/\sim)$, which can be readily proved to be continuous, uniquely extend to suitable

$$\begin{aligned}
|\cdot|&:\,L^2(T^*\mathrm{X})\to L^2(\mathfrak{m}),\\
\cdot&:\,L^\infty(\mathfrak{m})\times L^2(T^*\mathrm{X})\to L^2(T^*\mathrm{X}),
\end{aligned}$$

which endow $L^2(T^*\mathrm{X})$ with the structure of an $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module.

Finally, let us define the differential operator $\mathrm{d}:\mathrm{S}^2(\mathrm{X})\to L^2(T^*\mathrm{X})$ as $\mathrm{d}f:=[\mathrm{X},f]$ for every $f\in\mathrm{S}^2(\mathrm{X})$, where we think of $\mathsf{Pcm}/\sim$ as a subset of $L^2(T^*\mathrm{X})$. Note that

$$\mathrm{d}(\alpha\,f+\beta\,g)=[\mathrm{X},\alpha\,f+\beta\,g]=\alpha\,[\mathrm{X},f]+\beta\,[\mathrm{X},g]=\alpha\,\mathrm{d}f+\beta\,\mathrm{d}g\quad\forall f,g\in\mathrm{S}^2(\mathrm{X}),\ \alpha,\beta\in\mathbb{R},$$

proving that d is a linear map. Also $|\mathrm{d}f|=\big|[\mathrm{X},f]\big|=|Df|$ holds $\mathfrak{m}$-a.e. for any $f\in\mathrm{S}^2(\mathrm{X})$, which shows the validity of i). To conclude, observe that the family of all finite sums of the form $\sum_{i=1}^n\chi_{E_i}\mathrm{d}f_i$, with $(E_i)_{i=1}^n$ Borel partition of X and $(f_i)_{i=1}^n\subseteq\mathrm{S}^2(\mathrm{X})$, coincides with the space $\mathsf{Pcm}/\sim$, thus it is dense in $L^2(T^*\mathrm{X})$ by the very definition of $L^2(T^*\mathrm{X})$, proving ii) and accordingly the thesis. □

Theorem 13.3 (Closure of the differential) *Let $(f_n)_n\subseteq\mathrm{S}^2(\mathrm{X})$ be a sequence that pointwise converges $\mathfrak{m}$-a.e. to some limit function f. Suppose that $\mathrm{d}f_n\rightharpoonup\omega$ weakly in $L^2(T^*\mathrm{X})$ for some $\omega\in L^2(T^*\mathrm{X})$. Then $f\in\mathrm{S}^2(\mathrm{X})$ and $\mathrm{d}f=\omega$.*

Moreover, the same conclusion holds if $(f_n)_n\subseteq W^{1,2}(\mathrm{X})$ satisfies $f_n\rightharpoonup f$ and $\mathrm{d}f_n\rightharpoonup\omega$ weakly in $L^2(\mathfrak{m})$ and $L^2(T^*\mathrm{X})$, respectively.

Proof. By Mazur's lemma (recall Theorem 10.1) we can assume wlog that $\mathrm{d}f_n\to\omega$ in the strong topology of $L^2(T^*\mathrm{X})$. In particular, $|Df_n|=|\mathrm{d}f_n|\to|\omega|$ strongly in $L^2(\mathfrak{m})$ as $n\to\infty$, whence we have that $f \in \mathrm{S}^2(\mathrm{X})$ by Proposition 4.11. Moreover, it holds that

$$\begin{aligned} \varlimsup_{n\to\infty} \big\|\mathrm{d}f - \mathrm{d}f_n\big\|_{L^2(T^*\mathrm{X})} &\leq \varlimsup_{n\to\infty} \lim_{k\to\infty} \big\|\mathrm{d}(f_k - f_n)\big\|_{L^2(T^*\mathrm{X})} \\ &= \varlimsup_{n\to\infty} \lim_{k\to\infty} \big\||\mathrm{d}(f_k - f_n)|\big\|_{L^2(\mathfrak{m})} = 0, \end{aligned}$$

so that $\mathrm{d}f = \omega$ as required. Finally, the last statement follows from the first one by applying twice Mazur's lemma and by recalling that any strongly converging sequence in $L^2(\mathfrak{m})$ has a subsequence that is $\mathfrak{m}$-a.e. convergent to the same limit. □

Remark 13.4 We point out that the map

$$\begin{aligned} W^{1,2}(\mathrm{X}) &\longrightarrow L^2(\mathfrak{m}) \times L^2(T^*\mathrm{X}), \\ f &\longmapsto (f, \mathrm{d}f), \end{aligned} \tag{13.4}$$

is a linear isometry, as soon as the target space $L^2(\mathfrak{m}) \times L^2(T^*\mathrm{X})$ is endowed with the product norm $\big\|(f,\omega)\big\|^2 := \|f\|^2_{L^2(\mathfrak{m})} + \|\omega\|^2_{L^2(T^*\mathrm{X})}$. ■

14 Lesson [18/12/2017]

Theorem 14.1 (Calculus rules for the differential) *The following hold:*

A) LOCALITY. *Let $f, g \in \mathrm{S}^2(\mathrm{X})$ be given. Then $\mathrm{d}f = \mathrm{d}g$ holds $\mathfrak{m}$-a.e. in $\{f = g\}$.*

B) CHAIN RULE. *Let $f \in \mathrm{S}^2(\mathrm{X})$ be given.*

   B1) *If a Borel set $N \subseteq \mathbb{R}$ is $\mathcal{L}^1$-negligible, then $\mathrm{d}f = 0$ holds $\mathfrak{m}$-a.e. in $f^{-1}(N)$.*

   B2) *If $I \subseteq \mathbb{R}$ is an interval satisfying $(f_*\mathfrak{m})(\mathbb{R} \setminus I) = 0$ and $\varphi : I \to \mathbb{R}$ is a Lipschitz function, then $\varphi \circ f \in \mathrm{S}^2(\mathrm{X})$ and $\mathrm{d}(\varphi \circ f) = \varphi' \circ f \, \mathrm{d}f$. The expression $\varphi' \circ f \, \mathrm{d}f$ is a well-defined element of $L^2(T^*\mathrm{X})$ by* B1)*.*

C) LEIBNIZ RULE. *Let $f, g \in \mathrm{S}^2(\mathrm{X}) \cap L^\infty(\mathfrak{m})$ be given. Then $fg \in \mathrm{S}^2(\mathrm{X}) \cap L^\infty(\mathfrak{m})$ and it holds that $\mathrm{d}(fg) = f\,\mathrm{d}g + g\,\mathrm{d}f$.*

Proof. A) Note that $|\mathrm{d}f - \mathrm{d}g| = |D(f-g)| = 0$ holds $\mathfrak{m}$-a.e. in $\{f - g = 0\}$ by Theorem 8.7, whence $\mathrm{d}f = \mathrm{d}g$ holds $\mathfrak{m}$-a.e. in $\{f = g\}$, as required.
B1) We have that $|\mathrm{d}f| = |Df| = 0$ holds $\mathfrak{m}$-a.e. on $f^{-1}(N)$ by Theorem 8.7, so that $\mathrm{d}f = 0$ holds $\mathfrak{m}$-a.e. on $f^{-1}(N)$.
B2) The Lipschitz function $\varphi : I \to \mathbb{R}$ can be extended to a Lipschitz function $\overline{\varphi} : \mathbb{R} \to \mathbb{R}$ and the precise choice of such extension is irrelevant for the thesis to hold because $f^{-1}(\mathbb{R} \setminus I)$ has null $\mathfrak{m}$-measure. Then assume wlog $I = \mathbb{R}$. We know that $\varphi \circ f \in \mathrm{S}^2(\mathrm{X})$ by Theorem 8.7.

If φ is a linear function, then the chain rule just reduces to the linearity of the differential. If φ is an affine function, say that $\varphi(t) = at + b$, then $\mathrm{d}(\varphi \circ f) = \mathrm{d}(af + b) = a\,\mathrm{d}f = \varphi' \circ f \, \mathrm{d}f$. Now suppose that φ is a piecewise affine function. Say that $(I_n)_n$ is a sequence of intervals whose union covers the whole real line $\mathbb{R}$ and that $(\psi_n)_n$ is a sequence of affine functions such that $\varphi|_{I_n} = \psi_n$ holds for every $n \in \mathbb{N}$. Since φ' and ψ'_n coincide $\mathcal{L}^1$-a.e. in the interior of I_n, we have that $\mathrm{d}(\varphi \circ f) = \mathrm{d}(\psi_n \circ f) = \psi'_n \circ f \, \mathrm{d}f = \varphi' \circ f \, \mathrm{d}f$ holds $\mathfrak{m}$-a.e. on $f^{-1}(I_n)$ for all n, so that $\mathrm{d}(\varphi \circ f) = \varphi' \circ f \, \mathrm{d}f$ is verified $\mathfrak{m}$-a.e. on $\bigcup_n f^{-1}(I_n) = \mathrm{X}$, as required.

To prove the case of a general Lipschitz function $\varphi : \mathbb{R} \to \mathbb{R}$, we want to approximate φ with a sequence of piecewise affine functions: for any $n \in \mathbb{N} \setminus \{0\}$, let us denote by φ_n the function that coincides with φ at $\{i/n : i \in \mathbb{Z}\}$ and is affine in the interval $\big[i/n, (i+1)/n\big]$ for every $i \in \mathbb{Z}$. One can readily prove that $\mathrm{Lip}(\varphi_n) \leq \mathrm{Lip}(\varphi)$ for all n. Given any $i \in \mathbb{Z}$, we deduce from the fact that $\varphi'_n(t) = \fint_{i/n}^{(i+1)/n} \varphi' \, \mathrm{d}\mathcal{L}^1$ holds for all $t \in \big[i/n, (i+1)/n\big]$ and from an application of Jensen's inequality that

$$
\begin{aligned}
\int_{i/n}^{(i+1)/n} |\varphi'_n|^2 \, \mathrm{d}\mathcal{L}^1 = \frac{1}{n} \left| \fint_{i/n}^{(i+1)/n} \varphi' \, \mathrm{d}\mathcal{L}^1 \right|^2 &\leq \frac{1}{n} \fint_{i/n}^{(i+1)/n} |\varphi'|^2 \, \mathrm{d}\mathcal{L}^1 \\
&= \int_{i/n}^{(i+1)/n} |\varphi'|^2 \, \mathrm{d}\mathcal{L}^1.
\end{aligned}
\tag{14.1}
$$

Now fix $m \in \mathbb{N}$. It can be readily checked that $\varphi_n \to \varphi$ strongly in $L^2(-m, m)$, while (14.1) grants that $\int_{-m}^{m} |\varphi'_n|^2 \, \mathrm{d}\mathcal{L}^1 \leq \int_{-m}^{m} |\varphi'|^2 \, \mathrm{d}\mathcal{L}^1$ for every n, whence there is a subsequence $(n_k)_k$ such that $\varphi'_{n_k} \rightharpoonup g$ weakly in $L^2(-m, m)$ for some $g \in L^2(-m, m)$. This forces $g = \varphi'|_{(-m,m)}$, so that the original sequence $(\varphi'_n)_n$ satisfies $\varphi'_n \rightharpoonup \varphi'$ weakly in $L^2(-m, m)$. Moreover, it holds that $\int_{-m}^{m} |\varphi'|^2 \, \mathrm{d}\mathcal{L}^1 \leq \underline{\lim}_n \int_{-m}^{m} |\varphi'_n|^2 \, \mathrm{d}\mathcal{L}^1 \leq \int_{-m}^{m} |\varphi'|^2 \, \mathrm{d}\mathcal{L}^1$, thus necessarily $\varphi'_n \to \varphi'$ strongly in $L^2(-m, m)$. In particular, there exists a subsequence $(n_k)_k$ such that $\varphi'_{n_k}(t) \to \varphi'(t)$ for a.e. $t \in (-m, m)$. Up to performing a diagonalisation argument, we can therefore build a sequence $(\varphi_n)_n$ such that

$$
\begin{aligned}
&(\varphi_n)_n \subseteq \mathrm{LIP}(\mathbb{R}) \text{ are piecewise affine functions with } \sup_{n \in \mathbb{N}} \mathrm{Lip}(\varphi_n) \leq \mathrm{Lip}(\varphi), \\
&\varphi_n(t) \to \varphi(t) \text{ for every } t \in \mathbb{R} \text{ and } \varphi'_n(t) \to \varphi'(t) \text{ for } \mathcal{L}^1\text{-a.e. } t \in \mathbb{R}.
\end{aligned}
\tag{14.2}
$$

Finally, notice that $\int |\varphi'_n - \varphi'|^2 \circ f \, |\mathrm{d}f|^2 \, \mathrm{d}\mathfrak{m} \to 0$ by (14.2), by B1) and by an application of the dominated convergence theorem, in other words $\varphi'_n \circ f \, \mathrm{d}f \to \varphi' \circ f \, \mathrm{d}f$ in the strong topology of $L^2(T^*\mathrm{X})$. Since (14.2) also grants that $\varphi_n \circ f \to \varphi \circ f$ pointwise $\mathfrak{m}$-a.e. in X and since $\mathrm{d}(\varphi_n \circ f) = \varphi'_n \circ f \, \mathrm{d}f$ by the previous part of the proof, we deduce from Theorem 13.3 that $\mathrm{d}(\varphi_n \circ f) \to \varphi' \circ f \, \mathrm{d}f$ in $L^2(T^*\mathrm{X})$, thus accordingly $\mathrm{d}(\varphi \circ f) = \varphi' \circ f \, \mathrm{d}f$.
C) We already know that $fg \in \mathrm{S}^2(\mathrm{X}) \cap L^\infty(\mathfrak{m})$ by Theorem 8.7. In the case in which $f, g \geq 1$, we deduce from property B2) that

$$
\frac{\mathrm{d}(fg)}{fg} = \mathrm{d}\log(fg) = \mathrm{d}\big(\log(f) + \log(g)\big) = \mathrm{d}\log(f) + \mathrm{d}\log(g) = \frac{\mathrm{d}f}{f} + \frac{\mathrm{d}g}{g},
$$

whence we get $\mathrm{d}(fg) = f \, \mathrm{d}g + g \, \mathrm{d}f$ by multiplying both sides by fg.

In the general case $f, g \in L^\infty(\mathfrak{m})$, choose a constant $C > 0$ so big that $f + C, g + C \geq 1$.

By the previous case, we know that

$$\begin{aligned}\mathrm{d}\big((f+C)(g+C)\big) &= (f+C)\,\mathrm{d}(g+C)+(g+C)\,\mathrm{d}(f+C)\\ &= (f+C)\,\mathrm{d}g+(g+C)\,\mathrm{d}f\\ &= f\,\mathrm{d}g+g\,\mathrm{d}f+C\,\mathrm{d}(f+g),\end{aligned} \tag{14.3}$$

while a direct computation yields

$$\mathrm{d}\big((f+C)(g+C)\big)=\mathrm{d}\big(fg+C(f+g)+C^2\big)=\mathrm{d}(fg)+C\,\mathrm{d}(f+g). \tag{14.4}$$

By subtracting (14.4) from (14.3), we finally get that $\mathrm{d}(fg)=f\,\mathrm{d}g+g\,\mathrm{d}f$, as required. Hence the thesis is achieved. □

Proposition 14.2 *The set* $\{\mathrm{d}f\,:\,f\in W^{1,2}(\mathrm{X})\}$ *generates the tangent module* $L^2(T^*\mathrm{X})$.

Proof. Denote by $\mathscr{M}$ the module generated by $\{\mathrm{d}f\,:\,f\in W^{1,2}(\mathrm{X})\}$. It clearly suffices to prove that $\mathrm{d}f\in\mathscr{M}$ whenever $f\in\mathrm{S}^2(\mathrm{X})$. Fix any $\bar{x}\in\mathrm{X}$. For any $n,m\in\mathbb{N}$, let us call

$$\begin{aligned}f_n &:= (f\vee n)\wedge(-n)\in L^\infty(\mathfrak{m}),\\ \eta_m &:= \big(1-\mathsf{d}\big(\cdot,B_m(\bar{x})\big)\big)^+,\\ f_{nm} &:= \eta_m\,f_n\in L^2(\mathfrak{m}).\end{aligned}$$

Since the function f_n can be written as $\varphi_n\circ f$, where φ_n is the 1-Lipschitz function defined by $\varphi_n(t):=(t\vee n)\wedge(-n)$, we have that $f_n\in\mathrm{S}^2(\mathrm{X})$ by property B2) of Theorem 14.1, thus accordingly $f_{nm}\in W^{1,2}(\mathrm{X})$ by property C) of Theorem 14.1. More precisely, it holds that

$$\begin{aligned}\mathrm{d}f_n &= \varphi_n'\circ f\,\mathrm{d}f=\chi_{\{|f|\leq n\}}\,\mathrm{d}f,\\ \chi_{B_m(\bar{x})}\,\mathrm{d}f_{nm} &= \chi_{B_m(\bar{x})}\,\big(\eta_m\,\mathrm{d}f_n+f_n\,\mathrm{d}\eta_m\big)=\chi_{B_m(\bar{x})}\,\mathrm{d}f_n,\end{aligned}$$

so that $\mathrm{d}f=\mathrm{d}f_{nm}$ holds $\mathfrak{m}$-a.e. in $A_{nm}:=f^{-1}\big([-n,n]\big)\cap B_m(\bar{x})$. Given that $\mathfrak{m}(\mathrm{X}\setminus A_{nm})\searrow 0$ as $n,m\to\infty$, we deduce from the dominated convergence theorem that $\chi_{A_{nm}}\,\mathrm{d}f_{nm}\to\mathrm{d}f$ in the strong topology of $L^2(T^*\mathrm{X})$ as $n,m\to\infty$. Since each $\chi_{A_{nm}}\,\mathrm{d}f_{nm}$ belongs to $\mathscr{M}$, we conclude that $\mathrm{d}f\in\mathscr{M}$ as well. This proves the thesis. □

Proposition 14.3 *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a metric measure space. Then there exists a unique (up to unique isomorphism) couple* $(\mathscr{M},\widetilde{\mathrm{d}})$*, where* $\mathscr{M}$ *is a module and* $\widetilde{\mathrm{d}}:W^{1,2}(\mathrm{X})\to\mathscr{M}$ *is a linear map, such that* $|\widetilde{\mathrm{d}}f|=|Df|$ *holds* $\mathfrak{m}$*-a.e. for every* $f\in W^{1,2}(\mathrm{X})$ *and such that* $\mathscr{M}$ *is generated by* $\{\mathrm{d}f\,:\,f\in W^{1,2}(\mathrm{X})\}$.

Moreover, given any such couple there exists a unique map $\Psi:\mathscr{M}\to L^2(T^*\mathrm{X})$*, which is a module isomorphism, such that*

$$\begin{array}{ccc} W^{1,2}(\mathrm{X}) & \xrightarrow{\ \widetilde{\mathrm{d}}\ } & \mathscr{M}\\ \big\downarrow & & \big\downarrow \Psi\\ \mathrm{S}^2(\mathrm{X}) & \xrightarrow[\ \mathrm{d}\ ]{} & L^2(T^*\mathrm{X})\end{array} \tag{14.5}$$

is a commutative diagram.

Proof. EXISTENCE. One can repeat verbatim the proof of the existence part of Theorem 13.2. Otherwise, call $\mathscr{M}$ the submodule of $L^2(T^*\mathrm{X})$ that is generated by $\big\{\mathrm{d}f\,:\,f\in W^{1,2}(\mathrm{X})\big\}$ and define $\widetilde{\mathrm{d}}:=\mathrm{d}_{|W^{1,2}(\mathrm{X})}$. It can be easily seen that $(\mathscr{M},\widetilde{\mathrm{d}})$ satisfies the required properties.
UNIQUENESS. In order to get uniqueness, it is clearly enough to prove the last part of the statement. By the very same arguments that had been used in the proof of the uniqueness part of Theorem 13.2, one can see that the requirement that Ψ is an $L^\infty(\mathfrak{m})$-linear operator satisfying $\Psi(\widetilde{\mathrm{d}}f)=\mathrm{d}f$ for any $f\in W^{1,2}(\mathrm{X})$ forces a unique choice of $\Psi:\mathscr{M}\to L^2(T^*\mathrm{X})$. The surjectivity of Ψ stems from Proposition 14.2. $\square$

Proposition 14.4 *Fix $d\in\mathbb{N}\setminus\{0\}$. Let $L^2\big(\mathbb{R}^d,(\mathbb{R}^d)^*,\mathcal{L}^d\big)$ denote the space of all the $L^2(\mathcal{L}^d)$ 1-forms in $\mathbb{R}^d$. Let $\bar{\mathrm{d}}:W^{1,2}(\mathbb{R}^d)\to L^2\big(\mathbb{R}^d,(\mathbb{R}^d)^*,\mathcal{L}^d\big)$ be the map assigning to each Sobolev function $f\in W^{1,2}(\mathbb{R}^d)$ its distributional differential. Then*

$$\Big(L^2\big(\mathbb{R}^d,(\mathbb{R}^d)^*,\mathcal{L}^d\big),\bar{\mathrm{d}}\Big)\sim\big(L^2(T^*\mathbb{R}^d),\mathrm{d}\big),\tag{14.6}$$

in the sense that there exists a unique module isomorphism $\Phi:\,L^2(T^*\mathbb{R}^d)\to L^2\big(\mathbb{R}^d,(\mathbb{R}^d)^*,\mathcal{L}^d\big)$ such that $\Phi\circ\mathrm{d}=\bar{\mathrm{d}}$.

Proof. We know by Theorem 9.5 that $|\bar{\mathrm{d}}f|=|Df|$ holds $\mathcal{L}^d$-a.e. for every $f\in W^{1,2}(\mathbb{R}^d)$. Moreover, for any bounded Borel subset B of X and any $\omega\in(\mathbb{R}^d)^*$, there exists (by a cut-off argument) a function $f\in W^{1,2}(\mathbb{R}^d)$ such that $\bar{\mathrm{d}}f=\omega$ holds $\mathcal{L}^d$-a.e. in B. Hence the normed module $L^2\big(\mathbb{R}^d,(\mathbb{R}^d)^*,\mathcal{L}^d\big)$ is generated by the elements of the form $\chi_B\,\omega$. We thus conclude by applying Proposition 14.3. $\square$

15 Lesson [20/12/2017]

Let us denote by $L^0(\mathfrak{m})$ the vector space of all Borel functions $f:\mathrm{X}\to\mathbb{R}$, which are considered modulo $\mathfrak{m}$-a.e. equality. Then $L^0(\mathfrak{m})$ becomes a topological vector space when endowed with the following distance: choose any Borel probability measure $\mathfrak{m}'\in\mathscr{P}(\mathrm{X})$ such that $\mathfrak{m}\ll\mathfrak{m}'\ll\mathfrak{m}$ (for instance, pick any Borel partition $(E_n)_n$ made of sets having finite positive $\mathfrak{m}$-measure and call $\mathfrak{m}':=\sum_n\frac{\chi_{E_n}\mathfrak{m}}{2^n\,\mathfrak{m}(E_n)}$) and define

$$\mathsf{d}_{L^0}(f,g):=\int|f-g|\wedge 1\,\mathrm{d}\mathfrak{m}'\qquad\text{for every }f,g\in L^0(\mathfrak{m}).\tag{15.1}$$

Such distance may depend on the choice of $\mathfrak{m}'$, but its induced topology does not, as we are going to show in the next result:

Proposition 15.1 *A sequence $(f_n)_n\subseteq L^0(\mathfrak{m})$ is d_{L^0}-Cauchy if and only if*

$$\varlimsup_{n,m\to\infty}\mathfrak{m}\Big(E\cap\big\{|f_n-f_m|>\varepsilon\big\}\Big)=0\qquad\begin{array}{l}\textit{for every }\varepsilon>0\textit{ and }E\subseteq\mathrm{X}\\ \textit{Borel with }\mathfrak{m}(E)<+\infty.\end{array}\tag{15.2}$$

Proof. Necessity. Suppose that (15.2) holds. Fix $\varepsilon > 0$. Choose any point $\bar{x} \in \mathrm{X}$, then there exists $R > 0$ such that $\mathfrak{m}'\big(B_R(\bar{x})\big) \geq 1 - \varepsilon$. Recall that $\mathfrak{m}$ is finite on bounded sets by hypothesis, so that $\mathfrak{m}\big(B_R(\bar{x})\big) < +\infty$. Moreover, since $\mathfrak{m}'$ is a finite measure, we clearly have that $\chi_{B_R(\bar{x})} \frac{\mathrm{d}\mathfrak{m}'}{\mathrm{d}\mathfrak{m}} \in L^1(\mathfrak{m})$. Now let us call $A_{nm}(\varepsilon)$ the set $B_R(\bar{x}) \cap \big\{|f_n - f_m| > \varepsilon\big\}$. Then property (15.2) grants that $\chi_{A_{nm}(\varepsilon)} \to 0$ in $L^1(\mathfrak{m})$ as $n, m \to \infty$, whence an application of the dominated convergence theorem yields

$$\varlimsup_{n,m\to\infty} \mathfrak{m}'\big(A_{nm}(\varepsilon)\big) = \varlimsup_{n,m\to\infty} \int \chi_{A_{nm}(\varepsilon)}\, \chi_{B_R(\bar{x})}\, \frac{\mathrm{d}\mathfrak{m}'}{\mathrm{d}\mathfrak{m}}\, \mathrm{d}\mathfrak{m} = 0. \tag{15.3}$$

Therefore we deduce that

$$\begin{aligned}\int |f_n - f_m| \wedge 1 \, \mathrm{d}\mathfrak{m}' &= \int_{\mathrm{X}\setminus B_R(\bar{x})} |f_n - f_m| \wedge 1 \, \mathrm{d}\mathfrak{m}' + \int_{B_R(\bar{x})} |f_n - f_m| \wedge 1 \, \mathrm{d}\mathfrak{m}' \\ &\leq \varepsilon + \int_{B_R(\bar{x})\cap\{|f_n - f_m|\leq\varepsilon\}} |f_n - f_m| \wedge 1 \, \mathrm{d}\mathfrak{m}' + \int_{A_{nm}(\varepsilon)} |f_n - f_m| \wedge 1 \, \mathrm{d}\mathfrak{m}' \\ &\leq 2\,\varepsilon + \mathfrak{m}'\big(A_{nm}(\varepsilon)\big),\end{aligned}$$

from which we see that $\varlimsup_{n,m} \mathsf{d}_{L^0}(f_n, f_m) \leq 2\,\varepsilon$ by (15.3). By arbitrariness of $\varepsilon > 0$, we conclude that $\lim_{n,m} \mathsf{d}_{L^0}(f_n, f_m) = 0$, which shows that the sequence $(f_n)_n$ is d_{L^0}-Cauchy.
Sufficiency. Suppose that $(f_n)_n$ is d_{L^0}-Cauchy. Fix any $\varepsilon \in (0,1)$ and a Borel set $E \subseteq \mathrm{X}$ with $\mathfrak{m}(E) < +\infty$. Hence the Čebyšëv inequality yields

$$\mathfrak{m}'\big(\big\{|f_n - f_m| > \varepsilon\big\}\big) = \mathfrak{m}'\big(\big\{|f_n - f_m| \wedge 1 > \varepsilon\big\}\big) \leq \frac{1}{\varepsilon} \int |f_n - f_m| \wedge 1 \, \mathrm{d}\mathfrak{m}' = \frac{\mathsf{d}_{L^0}(f_n, f_m)}{\varepsilon},$$

so that $\varlimsup_{n,m} \mathfrak{m}'\big(\big\{|f_n - f_m| > \varepsilon\big\}\big) = 0$. Finally, observe that $\chi_E \frac{\mathrm{d}\mathfrak{m}}{\mathrm{d}\mathfrak{m}'} \in L^1(\mathfrak{m}')$, whence

$$\mathfrak{m}\Big(E \cap \big\{|f_n - f_m| > \varepsilon\big\}\Big) = \int \chi_E \, \frac{\mathrm{d}\mathfrak{m}}{\mathrm{d}\mathfrak{m}'}\, \chi_{\{|f_n - f_m|>\varepsilon\}}\, \mathrm{d}\mathfrak{m}' \overset{n,m}{\longrightarrow} 0$$

by dominated convergence theorem. Therefore (15.2) is proved. □

Remark 15.2 Recall that two metrizable spaces with the same Cauchy sequences have the same topology, while the converse implication does not hold in general. For instance, consider the real line $\mathbb{R}$ endowed with the following two distances:

$$\left.\begin{aligned}\mathsf{d}_1(x,y) &:= |x - y|, \\ \mathsf{d}_2(x,y) &:= \big|\arctan(x) - \arctan(y)\big|,\end{aligned}\right. \qquad \text{for every } x, y \in \mathbb{R}.$$

Then d_1 and d_2 induce the same topology on $\mathbb{R}$, but the d_2-Cauchy sequence $(x_n)_n \subseteq \mathbb{R}$ defined by $x_n := n$ is not d_1-Cauchy. ■

We now show that the distance d_{L^0} metrizes the *convergence in measure*:

Proposition 15.3 *Let $f \in L^0(\mathfrak{m})$ and $(f_n)_n \subseteq L^0(\mathfrak{m})$. Then the following are equivalent:*

i) *it holds that* $\mathsf{d}_{L^0}(f_n, f) \to 0$ *as* $n \to \infty$*,*

ii) *given any subsequence* $(n_m)_m$*, there exists a further subsequence* $(n_{m_k})_k$ *such that the limit* $\lim_k f_{n_{m_k}}(x) = f(x)$ *is verified for* $\mathfrak{m}$*-a.e.* $x \in \mathrm{X}$*,*

iii) *we have that* $\overline{\lim}_n \mathfrak{m}\big(E \cap \big\{|f_n - f| > \varepsilon\big\}\big) = 0$ *is satisfied for every* $\varepsilon > 0$ *and* $E \subseteq \mathrm{X}$ *Borel with* $\mathfrak{m}(E) < +\infty$*,*

iv) *we have that* $\overline{\lim}_n \mathfrak{m}'\big(\big\{|f_n - f| > \varepsilon\big\}\big) = 0$ *for every* $\varepsilon > 0$*.*

Proof. i) $\Longrightarrow$ ii) Since $|f_{n_m} - f| \wedge 1 \to 0$ in $L^1(\mathfrak{m}')$, there is $(n_{m_k})_k$ such that $|f_{n_{m_k}} - f|(x) \wedge 1 \to 0$ for $\mathfrak{m}'$-a.e. $x \in \mathrm{X}$, or equivalently $f_{n_{m_k}}(x) \to f(x)$ for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$.
ii) $\Longrightarrow$ iii) Fix $(n_m)_m$, $\varepsilon > 0$ and $E \subseteq \mathrm{X}$ Borel with $\mathfrak{m}(E) < +\infty$. Since $\chi_{\{|f_{n_{m_k}} - f| > \varepsilon\}} \to 0$ pointwise $\mathfrak{m}$-a.e. for some $(m_k)_k$ and $\chi_E \in L^1(\mathfrak{m})$, we apply the dominated convergence theorem to deduce that $\lim_k \int \chi_E \, \chi_{\{|f_{n_{m_k}} - f| > \varepsilon\}} \, \mathrm{d}\mathfrak{m} = 0$, i.e. $\lim_n \mathfrak{m}\big(E \cap \big\{|f_n - f| > \varepsilon\big\}\big) = 0$.
iii) $\Longrightarrow$ iv) Fix $\delta > 0$ and $\bar{x} \in \mathrm{X}$, then there is $R > 0$ such that $\mathfrak{m}'\big(\mathrm{X} \setminus B_R(\bar{x})\big) < \delta$. Exactly as we did in (15.3), we can prove that the fact that $\lim_n \mathfrak{m}\big(B_R(\bar{x}) \cap \big\{|f_n - f| > \varepsilon\big\}\big) = 0$ implies that $\lim_n \mathfrak{m}'\big(B_R(\bar{x}) \cap \big\{|f_n - f| > \varepsilon\big\}\big) = 0$ as well. Therefore

$$\overline{\lim_{n \to \infty}} \, \mathfrak{m}'\big(\big\{|f_n - f| > \varepsilon\big\}\big) \leq \delta + \overline{\lim_{n \to \infty}} \, \mathfrak{m}'\big(B_R(\bar{x}) \cap \big\{|f_n - f| > \varepsilon\big\}\big) = \delta.$$

By letting $\delta \searrow 0$, we thus conclude that $\overline{\lim}_n \mathfrak{m}'\big(\big\{|f_n - f| > \varepsilon\big\}\big) = 0$, as required.
iv) $\Longrightarrow$ i) Take any $\varepsilon \in (0, 1)$. Notice that

$$\begin{aligned}\mathsf{d}_{L^0}(f_n, f) = \int |f_n - f| \wedge 1 \, \mathrm{d}\mathfrak{m}' &= \int_{\{|f_n - f| \leq \varepsilon\}} |f_n - f| \wedge 1 \, \mathrm{d}\mathfrak{m}' + \int_{\{|f_n - f| > \varepsilon\}} |f_n - f| \wedge 1 \, \mathrm{d}\mathfrak{m}' \\ &\leq \varepsilon + \mathfrak{m}'\big(\big\{|f_n - f| > \varepsilon\big\}\big),\end{aligned}$$

whence $\overline{\lim}_n \mathsf{d}_{L^0}(f_n, f) \leq \varepsilon$, thus accordingly $\lim_n \mathsf{d}_{L^0}(f_n, f) = 0$ by arbitrariness of ε. $\square$

In particular, Proposition 15.3 grants that the completeness of $L^0(\mathfrak{m})$ cannot depend on the particular choice of the measure $\mathfrak{m}'$.

Remark 15.4 The inclusion map $L^p(\mathfrak{m}) \hookrightarrow L^0(\mathfrak{m})$ is continuous for every $p \in [1, \infty]$.

Indeed, choose any $\mathfrak{m}' \in \mathscr{P}(\mathrm{X})$ with $\mathfrak{m} \ll \mathfrak{m}' \leq \mathfrak{m}$ and define d_{L^0} as in (15.1). Now take any sequence $(f_n)_n$ in $L^p(\mathfrak{m})$ that $L^p(\mathfrak{m})$-converges to some limit $f \in L^p(\mathfrak{m})$. In particular, we have that $f_n \to f$ in $L^p(\mathfrak{m}')$, so that

$$\mathsf{d}_{L^0}(f_n, f) = \int |f_n - f| \wedge 1 \, \mathrm{d}\mathfrak{m}' \leq \int |f_n - f| \, \mathrm{d}\mathfrak{m}' \leq \|f_n - f\|_{L^p(\mathfrak{m}')} \xrightarrow{n} 0,$$

which proves the claim. $\blacksquare$

Exercise 15.5 Prove that $L^p(\mathfrak{m})$ is dense in $L^0(\mathfrak{m})$ for every $p \in [1, \infty]$.

Proposition 15.6 *The space* $\big(L^0(\mathfrak{m}), \mathsf{d}_{L^0}\big)$ *is complete and separable.*

Proof. COMPLETENESS. Fix a d_{L^0}-Cauchy sequence $(f_n)_n \subseteq L^0(\mathfrak{m})$ and some $\varepsilon > 0$. Then there exists a subsequence $(n_k)_k$ such that $\mathfrak{m}'\big(\{|f_{n_{k+1}} - f_{n_k}| > 1/2^k\}\big) < \varepsilon/2^k$ holds for all k. Call $A_k := \{|f_{n_{k+1}} - f_{n_k}| > 1/2^k\}$ and $A := \bigcup_k A_k$, so that $\mathfrak{m}'(A) \leq \varepsilon$. Given any $x \in \mathrm{X} \setminus A$, it holds that $\big|f_{n_{k+1}}(x) - f_{n_k}(x)\big| \leq 1/2^k$ for all k, in other words $\big(f_{n_k}(x)\big)_k \subseteq \mathbb{R}$ is a Cauchy (thus also converging) sequence, say $f_{n_k}(x) \to f(x)$ for some $f(x) \in \mathbb{R}$. Up to performing a diagonalisation argument, we have that $f_{n_k} \to f$ pointwise $\mathfrak{m}'$-a.e. for some $f \in L^0(\mathfrak{m})$. Therefore Proposition 15.3 grants that $\mathsf{d}_{L^0}(f_n, f) \to 0$, as required.
SEPARABILITY. Fix $f \in L^0(\mathfrak{m})$. Take any increasing sequence $(E_n)_n$ of Borel subsets of X having finite $\mathfrak{m}$-measure and such that $\mathrm{X} = \bigcup_n E_n$. Denote $f_n := \big((\chi_{E_n} f) \wedge n\big) \vee (-n)$ for every $n \in \mathbb{N}$. By dominated convergence theorem, we have that $f_n \to f$ in $L^0(\mathfrak{m})$. Moreover, it holds that $(f_n)_n \subseteq L^1(\mathfrak{m})$. Hence we get the thesis by recalling Remark 15.4, Exercise 15.5 and the fact that $L^1(\mathfrak{m})$ is separable. □

Remark 15.7 Notice that $\mathsf{d}_{L^0}(f,g) = \mathsf{d}_{L^0}(f+h, g+h)$ for every $f, g, h \in L^0(\mathfrak{m})$. However, the distance d_{L^0} is not induced by any norm, as shown by the fact that $\mathsf{d}_{L^0}(\lambda f, 0)$ differs from $|\lambda|\,\mathsf{d}_{L^0}(f,0)$ for some $\lambda \in \mathbb{R}$ and $f \in L^0(\mathfrak{m})$. ■

Exercise 15.8 Suppose that the measure $\mathfrak{m}$ has no atoms. Let $L:\ L^0(\mathfrak{m}) \to \mathbb{R}$ be linear and continuous. Then $L = 0$.

Definition 15.9 (L^0-normed module) *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space. We define an* $L^0(\mathfrak{m})$-normed module *as any quadruple* $\big(\mathscr{M}^0, \tau, \cdot, |\cdot|\big)$, *where*

i) $(\mathscr{M}^0, \tau)$ *is a topological vector space,*

ii) *the bilinear map* $\cdot:\ L^0(\mathfrak{m}) \times \mathscr{M}^0 \to \mathscr{M}^0$ *satisfies* $f \cdot (g \cdot v) = (fg) \cdot v$ *and* $\hat{1} \cdot v = v$ *for every* $f, g \in L^0(\mathfrak{m})$ *and* $v \in \mathscr{M}^0$,

iii) *the map* $|\cdot|:\ \mathscr{M}^0 \to L^0(\mathfrak{m})$, *which satisfies both* $|v| \geq 0$ *and* $|f \cdot v| = |f||v|$ $\mathfrak{m}$*-a.e. for every* $v \in \mathscr{M}^0$ *and* $f \in L^0(\mathfrak{m})$, *is such that the function* $\mathsf{d}_{\mathscr{M}^0}:\ \mathscr{M}^0 \times \mathscr{M}^0 \to [0, +\infty)$, *defined by*

$$\mathsf{d}_{\mathscr{M}^0}(v, w) := \int |v - w| \wedge 1 \, \mathrm{d}\mathfrak{m}' \quad \text{for some } \mathfrak{m}' \in \mathscr{P}(\mathrm{X}) \text{ with } \mathfrak{m} \ll \mathfrak{m}' \ll \mathfrak{m}, \tag{15.4}$$

is a complete distance on $\mathscr{M}^0$ *that induces the topology* τ.

Remark 15.10 The topology τ in the definition of an L^0-normed module does not depend on the particular choice of the measure $\mathfrak{m}'$. Indeed, it holds that a given sequence $(v_n)_n \subseteq \mathscr{M}^0$ is $\mathsf{d}_{\mathscr{M}^0}$-Cauchy if and only if

$$\varlimsup_{n,m\to\infty} \mathfrak{m}\Big(E \cap \big\{|v_n - v_m| > \varepsilon\big\}\Big) = 0 \qquad \begin{array}{l}\text{for every } \varepsilon > 0 \text{ and } E \subseteq \mathrm{X} \\ \text{Borel with } \mathfrak{m}(E) < +\infty.\end{array}$$

The previous statement can be achieved by arguing as in the proof of Proposition 15.1. ■

Definition 15.11 (**L^0-completion**) *Let $\mathscr{M}$ be an $L^2(\mathfrak{m})$-normed module. Then we define an $L^0(\mathfrak{m})$-*completion *of $\mathscr{M}$ as any couple $(\mathscr{M}^0, i)$, where $\mathscr{M}^0$ is an $L^0(\mathfrak{m})$-normed module and $i : \mathscr{M} \to \mathscr{M}^0$ is a linear operator with dense image that preserves the pointwise norm, i.e. such that the equality $\big|i(v)\big| = |v|$ holds $\mathfrak{m}$-a.e. for every $v \in \mathscr{M}$.*

Remark 15.12 Let $\mathscr{M}^0$ be an $L^0(\mathfrak{m})$-normed module. Then

$$\begin{aligned} |\cdot| :\ \mathscr{M}^0 \to L^0(\mathfrak{m}) \quad &\text{is continuous,} \\ \cdot :\ L^0(\mathfrak{m}) \times \mathscr{M}^0 \to \mathscr{M}^0 \quad &\text{is continuous.} \end{aligned} \tag{15.5}$$

To prove the first in (15.5), notice that $|v+w| \leq |v| + |w|$ holds $\mathfrak{m}$-a.e. for any $v, w \in \mathscr{M}^0$, so

$$\mathsf{d}_{L^0}\big(|v|, |w|\big) = \int \big||v| - |w|\big| \wedge 1 \,\mathrm{d}\mathfrak{m}' \leq \int |v-w| \wedge 1 \,\mathrm{d}\mathfrak{m}' = \mathsf{d}_{\mathscr{M}^0}(v, w).$$

To prove the second in (15.5), suppose that $f_n \to f$ and $v_n \to v$ in $L^0(\mathfrak{m})$ and $\mathscr{M}^0$, respectively. We aim to show that $f_n v_n \to f v$ in $\mathscr{M}^0$. First of all, observe that

$$|f_n v_n - f v| \leq |f_n||v_n - v| + |v||f_n - f| \quad \text{holds } \mathfrak{m}\text{-a.e. in X.} \tag{15.6}$$

We claim that

$$\forall \delta > 0 \quad \exists\, M > 0 : \quad \varlimsup_{n\to\infty} \mathfrak{m}'\big(\big\{|f_n| > M\big\}\big) < \delta. \tag{15.7}$$

Clearly, given any $\delta > 0$ there exists $M > 1$ such that $\mathfrak{m}'\big(\big\{|f| > M-1\big\}\big) < \delta$. Hence

$$\varlimsup_{n\to\infty} \mathfrak{m}'\big(\big\{|f_n| > M\big\}\big) \leq \mathfrak{m}'\big(\big\{|f| > M-1\big\}\big) + \varlimsup_{n\to\infty} \mathfrak{m}'\big(\big\{|f_n - f| > 1\big\}\big) < \delta,$$

which proves (15.7). Now let $\varepsilon > 0$ be fixed. Given any $\delta > 0$, take $M > 0$ as in (15.7), so

$$\varlimsup_{n} \mathfrak{m}'\big(\big\{|f_n||v_n - v| > \varepsilon/2\big\}\big) \leq \varlimsup_{n} \mathfrak{m}'\big(\big\{|f_n| > M\big\}\big) + \varlimsup_{n} \mathfrak{m}'\big(\big\{|v_n - v| > \varepsilon/(2M)\big\}\big) < \delta.$$

Hence $\varlimsup_n \mathfrak{m}'\big(\big\{|f_n||v_n - v| > \varepsilon/2\big\}\big) = 0$ by letting $\delta \searrow 0$. In an analogous way, we can see that also $\varlimsup_n \mathfrak{m}'\big(\big\{|v||f_n - f| > \varepsilon/2\big\}\big) = 0$. Therefore (15.6) yields

$$\varlimsup_{n} \mathfrak{m}'\big(\big\{|f_n v_n - f v| > \varepsilon\big\}\big) \leq \varlimsup_{n} \mathfrak{m}'\big(\big\{|f_n||v_n - v| > \varepsilon/2\big\}\big) + \varlimsup_{n} \mathfrak{m}'\big(\big\{|v||f_n - f| > \varepsilon/2\big\}\big) = 0,$$

which proves that $f_n v_n \to f v$ in $\mathscr{M}^0$, as desired. ■

Proposition 15.13 (Existence and uniqueness of the L^0-completion) *Let $\mathscr{M}$ be any given $L^2(\mathfrak{m})$-normed module. Then there exists a unique $L^0(\mathfrak{m})$-completion $(\mathscr{M}^0, i)$ of $\mathscr{M}$.*

Uniqueness has to be intended up to unique isomorphism, in the following sense: given any other $L^0(\mathfrak{m})$-completion $(\widetilde{\mathscr{M}}^0, \tilde{i})$ of $\mathscr{M}$, there is a unique module isomorphism $\Psi : \mathscr{M}^0 \to \widetilde{\mathscr{M}}^0$ such that

$$\begin{tikzcd} \mathscr{M} \arrow[r, "i"] \arrow[dr, "\tilde{i}"'] & \mathscr{M}^0 \arrow[d, "\Psi"] \\ & \widetilde{\mathscr{M}}^0 \end{tikzcd} \tag{15.8}$$

is a commutative diagram. Moreover, it holds that

i) *the map* $i : \mathscr{M} \to \mathscr{M}^0$ *is continuous and* $i(fv) = f\,i(v)$ *for all* $f \in L^\infty(\mathfrak{m})$ *and* $v \in \mathscr{M}$,

ii) $i(\mathscr{M})$ *coincides with the set of all* $v \in \mathscr{M}^0$ *such that* $|v| \in L^2(\mathfrak{m})$.

Proof. i) Since $\big|i(v)\big| = |v|$ holds $\mathfrak{m}$-a.e. for every $v \in \mathscr{M}$, we deduce that $\big\|\big|i(v)\big|\big\|_{L^2(\mathfrak{m})} = \|v\|_{\mathscr{M}}$ for every $v \in \mathscr{M}$. Hence if $(v_n)_n \subseteq \mathscr{M}$ converges to $v \in \mathscr{M}$ then $\big\|\big|i(v_n - v)\big|\big\|_{L^2(\mathfrak{m})} \to 0$, so that $\mathsf{d}_{\mathscr{M}^0}\big(i(v_n), i(v)\big) = \mathsf{d}_{L^0}\big(|i(v_n - v)|, 0\big) \to 0$ by Remark 15.4.

Moreover, we have that $\chi_E\, i(v) = i(\chi_E\, v)$ for every $E \subseteq \mathrm{X}$ Borel, indeed

$$\big|\chi_E\, i(v) - i(\chi_E\, v)\big| = \begin{cases} \big|i(v) - i(\chi_E\, v)\big| = \big|i((1-\chi_E)v)\big| = \chi_{E^c}|v| = 0 & \mathfrak{m}\text{-a.e. on } E, \\ \big|i(\chi_E\, v)\big| = |\chi_E\, v| = \chi_E|v| = 0 & \mathfrak{m}\text{-a.e. on } E^c. \end{cases}$$

By linearity of i, we immediately see that $f\,i(v) = i(fv)$ for any simple function $f : \mathrm{X} \to \mathbb{R}$, thus also for every $f \in L^\infty(\mathfrak{m})$ by continuity of i and Remark 15.12.

Uniqueness. The choice $\Psi\big(i(v)\big) := \widetilde{i}(v)$ for every $v \in \mathscr{M}$ is obliged. Moreover, we have that the equalities $\big|i(v)\big| = |v| = \big|\widetilde{i}(v)\big|$ hold $\mathfrak{m}$-a.e. in X for every $v \in \mathscr{M}$. Hence

$$\begin{aligned} \mathsf{d}_{\widetilde{\mathscr{M}}^0}\big(\Psi\big(i(v)\big), \Psi\big(i(w)\big)\big) &= \int \big|\widetilde{i}(v) - \widetilde{i}(w)\big| \wedge 1 \,\mathrm{d}\mathfrak{m}' = \int |v - w| \wedge 1 \,\mathrm{d}\mathfrak{m}' \\ &= \int \big|i(v) - i(w)\big| \wedge 1 \,\mathrm{d}\mathfrak{m}' = \mathsf{d}_{\mathscr{M}^0}\big(i(v), i(w)\big) \end{aligned}$$

is satisfied for every $v, w \in \mathscr{M}$, which shows that $\Psi : i(\mathscr{M}) \to \widetilde{i}(\mathscr{M})$ is an isometry, in particular it is continuous. Since $i(\mathscr{M})$ is dense in $\mathscr{M}^0$, we can uniquely extend Ψ to some map $\Psi : \mathscr{M}^0 \to \widetilde{\mathscr{M}}^0$, which is a linear isometry. Furthermore, Ψ preserves the pointwise norm and the multiplication by $L^0(\mathfrak{m})$-functions by i) and Remark 15.12, while it is surjective by density of $\widetilde{i}(\mathscr{M})$ in $\widetilde{\mathscr{M}}^0$. Therefore this (uniquely determined) map Ψ is module isomorphism satisfying property (15.8).

Existence. Define the distance d_0 on $\mathscr{M}$ as $\mathsf{d}_0(v, w) := \int |v - w| \wedge 1 \,\mathrm{d}\mathfrak{m}'$ and denote by $\mathscr{M}^0$ the completion of $(\mathscr{M}, \mathsf{d}_0)$. It can be readily proved that

$$\begin{aligned} \mathsf{d}_0(v_1 + w_1, v_2 + w_2) &\leq \mathsf{d}_0(v_1, v_2) + \mathsf{d}_0(w_1, w_2), \\ \mathsf{d}_0(\lambda\, v, \lambda\, w) &\leq \big(|\lambda| \vee 1\big)\, \mathsf{d}_0(v, w), \\ \mathsf{d}_{L^0}\big(|v|, |w|\big) &\leq \mathsf{d}_0(v, w), \\ (f_n)_n\ L^0(\mathfrak{m})\text{-Cauchy},\ (v_n)_n\ \mathsf{d}_0\text{-Cauchy} &\Longrightarrow\ (f_n v_n)_n\ \mathsf{d}_0\text{-Cauchy}. \end{aligned} \tag{15.9}$$

The first two properties in (15.9) grant that the vector space structure of $\mathscr{M}$ can be carried over to $\mathscr{M}^0$, while the third one and the fourth one show that we can extend to $\mathscr{M}^0$ the pointwise norm and the multiplication by $L^0(\mathfrak{m})$-functions, respectively.

ii) It clearly suffices to prove that $i(\mathscr{M}) \supseteq \big\{v \in \mathscr{M}^0 : |v| \in L^2(\mathfrak{m})\big\}$. To this aim, let us fix any $v \in \mathscr{M}^0$ with $|v| \in L^2(\mathfrak{m})$. There exists $(v_n)_n \subseteq \mathscr{M}$ such that $i(v_n) \to v$ in $\mathscr{M}^0$. Define

$$w_n := \chi_{\{|i(v_n)|>0\}} \frac{|v|}{\big|i(v_n)\big|}\, i(v_n) \in \mathscr{M}^0 \quad \text{for every } n \in \mathbb{N}.$$

Notice that $|w_n| = \chi_{\{|i(v_n)|>0\}}\, |v| \in L^2(\mathfrak{m})$ for every $n \in \mathbb{N}$. Moreover, one can easily prove that $(w_n)_n \subseteq i(\mathscr{M})$. Since $|w_n - v| \to 0$ in $L^2(\mathfrak{m})$ by dominated convergence theorem, we thus conclude that $v \in i(\mathscr{M})$ as well. □

16 Lesson [08/01/2018]

Proposition 16.1 *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space. Then there exists a unique (up to unique isomorphism) couple* $(\mathscr{M}^0, \mathrm{d}^0)$*, where* $\mathscr{M}^0$ *is an* L^0*-normed module and* $\mathrm{d}^0 : \mathrm{S}^2_{loc}(\mathrm{X}) \to \mathscr{M}^0$ *is a linear map, such that* $|\mathrm{d}^0 f| = |Df|$ *holds* $\mathfrak{m}$*-a.e. for every* $f \in \mathrm{S}^2_{loc}(\mathrm{X})$ *and such that* L^0*-linear combinations of elements in* $\left\{\mathrm{d}^0 f \,:\, f \in \mathrm{S}^2_{loc}(\mathrm{X})\right\}$ *are dense in* $\mathscr{M}^0$*.*

Moreover, given any such couple there exists a unique map $\iota : L^2(T^*\mathrm{X}) \to \mathscr{M}^0$*, which is* L^∞*-linear, continuous, preserving the pointwise norm such that*

$$\begin{array}{ccc} W^{1,2}(\mathrm{X}) & \xrightarrow{\ \mathrm{d}\ } & L^2(T^*\mathrm{X}) \\ \Big\downarrow & & \Big\downarrow \iota \\ \mathrm{S}^2_{loc}(\mathrm{X}) & \xrightarrow[\ \mathrm{d}^0\ ]{} & \mathscr{M}^0 \end{array} \tag{16.1}$$

is a commutative diagram. Moreover, the image of $L^2(T^*\mathrm{X})$ *in* $\mathcal{M}^0$ *via* ι *is dense.*

Proof. Uniqueness Follows along the same lines of Theorem 13.2. For Existence we consider the L^0-completion $(\mathscr{M}^0, \iota)$ of $L^2(T^*\mathrm{X})$ and recall that for any $f \in \mathrm{S}^2_{loc}(\mathrm{X})$ there is a partition (E_n) of X and functions $f_n \in W^{1,2}(\mathrm{X})$ such that $f = f_n$ $\mathfrak{m}$-a.e. on E_n for every $n \in \mathbb{N}$. It is clear that the series $\sum_n \chi_{E_n} \iota(\mathrm{d} f_n)$ converges in $\mathscr{M}^0$ and the locality of the differential grants that its limit, which we shall call $\mathrm{d}^0 f$, does not depend on the particular choice of $(E_n), (f_n)$.

Then the identity $|\mathrm{d}^0 f| = |Df|$ follows from the construction and the analogous property of the differential. Also, we know that L^∞-linear combinations of $\{\mathrm{d} f : f \in W^{1,2}(\mathrm{X})\}$ are dense in $L^2(T^*\mathrm{X})$ and that $\iota(L^2(T^*\mathrm{X}))$ is dense in $\mathscr{M}^0$. Thus L^∞-linear combinations of $\{\iota(\mathrm{d} f) = \mathrm{d}^0 f : f \in W^{1,2}(\mathrm{X})\}$ are dense in $\mathscr{M}^0$.

This construction also shows the existence and uniqueness of ι as in (16.1). $\square$

Lemma 16.2 (Essential supremum) *Let* $f_i : \mathrm{X} \to \mathbb{R} \cup \{\pm\infty\}$ *be given functions,* $i \in I$*. Then there exists a unique (up to equality* $\mathfrak{m}$*-a.e.) function* $g : \mathrm{X} \to \mathbb{R} \cup \{\pm\infty\}$ *such that*

i) $g \geq f_i$ $\mathfrak{m}$*-a.e. for every* $i \in I$*,*

ii) if $h \geq f_i$ $\mathfrak{m}$*-a.e. for every* $i \in I$ *then* $h \geq g$ $\mathfrak{m}$*-a.e..*

Moreover, there is an at most countable subfamily $(f_{i,n})$ *of* $(f_i)_{i \in I}$ *such that* $g = \sup_n f_{i_n}$*.*

Such g *is called* essential supremum *of the family* (f_i)*.*

Proof. The $\mathfrak{m}$-a.e. uniqueness of g follows trivially from (ii), so we pass to existence.

Replacing if necessary the f_i's with $\varphi \circ f_i$, where $\varphi : \mathbb{R} \cup \{\pm\infty\} \to [0,1]$ is monotone and injective, we can assume that the given functions are bounded. Similarly, replacing $\mathfrak{m}$ with a Borel probability measure with the same negligible sets we can assume that $\mathfrak{m}$ is a probability measure.

Now let

$$\mathcal{A} := \left\{ f_{i_1} \vee \ldots \vee f_{i_n} \;:\; n \in \mathbb{N},\ i_j \in I\ \forall j = 1, \ldots, n \right\}$$

put $S := \sup_{\tilde f \in \mathcal{A}} \int \tilde f \, \mathrm{d}\mathfrak{m}$ and notice that since the f_i's are uniformly bounded and $\mathfrak{m}(\mathrm{X}) < \infty$ we have $S < \infty$. Let $(\tilde f_n) \subset \mathcal{A}$ be such that $S = \sup_n \int \tilde f_n \, \mathrm{d}\mathfrak{m}$, put $g := \sup_n \tilde f_n$ so that by construction we have $S = \int g \, \mathrm{d}\mathfrak{m}$ and by definition there must exist a countable family $(f_{i,n})$, $i_n \in I$, such that $g = \sup_{n\in\mathbb{N}} f_{i_n}$.

We claim that g satisfies $(i), (ii)$. Indeed, suppose (i) does not hold, i.e. for some $\bar i \in I$ it hold $f_{\bar i} > g$ on a set of positive $\mathfrak{m}$-measure. Then $S = \int g \, \mathrm{d}\mathfrak{m} < \int g \vee f_{\bar i} \, \mathrm{d}\mathfrak{m} = \lim_{n\to\infty} \int f_{i_1} \vee \ldots \vee f_{i_n} \vee f_{\bar i} \, \mathrm{d}\mathfrak{m}$, contradicting the definition of S. For (ii) simply notice that if $h \geq f_{i_n}$ $\mathfrak{m}$-a.e. for every n, then it holds $h \geq g$ $\mathfrak{m}$-a.e.. □

We now define the concept of *dual* $\mathscr{M}^*$ of an L^2-normed L^∞-module $\mathscr{M}$. As a set we define

$$\mathscr{M}^* := \big\{ L : \mathscr{M} \to L^1(\mathrm{X}) \ : \text{ linear, continuous and s.t. } L(fv) = fL(v) \ \forall v \in \mathscr{M}, \ f \in L^\infty(\mathrm{X}) \big\}$$

and we endow it with the operator norm, i.e. $\|L\|_* := \sup_{\|v\|\leq 1} \|L(v)\|_{L^1(\mathrm{X})}$. The product of $f \in L^\infty(\mathrm{X})$ and $L \in \mathscr{M}^*$ is defined as

$$(fL)(v) := fL(v) \qquad \forall v \in \mathscr{M},$$

and the pointwise norm as

$$|L|_* := \operatorname*{ess\,sup}_{v\in\mathscr{M}, \ |v|\leq 1 \ \mathfrak{m}-a.e.} L(v)$$

Proposition 16.3 *The space $\mathscr{M}^*$ with the operations just defined is a L^2-normed L^∞-module and moreover it holds*

$$|L|_* = \operatorname*{ess\,sup}_{v\in\mathscr{M}, \ |v|\leq 1 \ \mathfrak{m}-a.e.} |L(v)|, \qquad \forall L \in \mathscr{M}^*, \tag{16.2a}$$

$$|L(v)| \leq |v||L|_* \quad \mathfrak{m}-a.e. \qquad \forall v \in \mathscr{M}, \ L \in \mathscr{M}^*. \tag{16.2b}$$

Proof. The fact that $(\mathscr{M}^*, \|\cdot\|_*)$ is a Banach space is obvious. The fact that $fL \in \mathscr{M}^*$ for $f \in L^\infty(\mathrm{X})$ and $L \in \mathscr{M}^*$ follows from the commutativity of $L^\infty(\mathrm{X})$: indeed, the fact that fL is linear and continuous are obvious and moreover we have

$$(fL)(gv) = fL(gv) = fgL(v) = gfL(v) = g(fL)(v).$$

The required properties of the multiplication by a L^∞-functions are easily derived, as for any $v \in \mathscr{M}$ we have

$$\big(f(gL)\big)(v) = f\big((gL)(v)\big) = f\big(gL(v)\big) = fgL(v) = (fgL)(v)$$

and $(\hat 1 L)(v) = L(\hat 1 v) = L(v)$. We come to the pointwise norm. To check that $|L|_* \geq 0$ pick $v = 0$ in the definition. Inequality $\leq$ in (16.2a) is obvious, for the converse let $v \in \mathscr{M}$ be with $|v| \leq 1$ $\mathfrak{m}$-a.e. and put $\tilde v := \chi_{\{L(v)\geq 0\}} v - \chi_{\{L(v)<0\}} v$, so that $|\tilde v| = |v|$ and $L(\tilde v) = |L(v)|$. Then it holds $|L|_* \geq L(\tilde v) = |L(v)|$, thus getting (16.2a).

We pass to (16.2b) and observe that $\chi_{\{v=0\}}L(v) = L(\chi_{\{v=0\}}v) = 0$, so that (16.2b) holds $\mathfrak{m}$-a.e. on $\{v = 0\}$. Hence it sufficient to prove that for any $c > 0$ the same inequality holds $\mathfrak{m}$-a.e. on $\{|v| \in [c, c^{-1}]\}$. To see this, notice that on $\{|v| \in [c, c^{-1}]\}$ the functions $|v|, |v|^{-1}$ are both in $L^\infty(\mathrm{X})$, hence we can write $\chi_{\{|v|\in[c,c^{-1}]\}}v = \chi_{\{|v|\in[c,c^{-1}]\}}|v|\frac{v}{|v|}$ and since $|\chi_{\{|v|\in[c,c^{-1}]\}}\frac{v}{|v|}| \leq 1$ $\mathfrak{m}$-a.e. we obtain

$$\chi_{\{|v|\in[c,c^{-1}]\}}|L(v)| = \chi_{\{|v|\in[c,c^{-1}]\}}\Big|L\big(|v|\frac{v}{|v|}\big)\Big| = \chi_{\{|v|\in[c,c^{-1}]\}}|v|\Big|L\big(\frac{v}{|v|}\big)\Big| \leq \chi_{\{|v|\in[c,c^{-1}]\}}|v|\,|L|_*.$$

We now observe that for every $f \in L^\infty(\mathrm{X})$ and $L \in \mathscr{M}^*$ we have

$$|fL|_* = \operatorname{ess\,sup}|fL(v)| = \operatorname{ess\,sup}|f||L(v)| = |f|\operatorname{ess\,sup}|L(v)| = |f||L|_*,$$

where all the essential supremum are taken among all $v \in \mathscr{M}$ with $|v| \leq 1$ $\mathfrak{m}$-a.e.. Hence to conclude we need to prove that

$$\|L\|_* = \sqrt{\int |L|_*^2 \,\mathrm{d}\mathfrak{m}}. \tag{16.3}$$

The inequality

$$\int |L(v)| \,\mathrm{d}\mathfrak{m} \leq \int |v||L|_* \,\mathrm{d}\mathfrak{m} \leq \sqrt{\int |v|^2 \,\mathrm{d}\mathfrak{m}}\sqrt{\int |L|_*^2 \,\mathrm{d}\mathfrak{m}} = \|v\|\sqrt{\int |L|_*^2 \,\mathrm{d}\mathfrak{m}}$$

valid for any $v \in \mathscr{M}$, $L \in \mathscr{M}^*$ shows that $\leq$ holds in (16.3). For the converse inequality recall that the properties of the essential supremum ensure that there are $(v_n) \subset \mathscr{M}$ with $|v_n| \leq 1$ $\mathfrak{m}$-a.e. for every $n \in \mathbb{N}$ such that $|L|_* = \sup_n L(v_n)$. Define recursively $(\tilde{v}_n) \subset \mathscr{M}$ by putting $\tilde{v}_0 := v_0$ and

$$\tilde{v}_{n+1} = \chi_{\{L(v_{n+1})\geq L(\tilde{v}_n)\}}v_{n+1} + \chi_{\{L(v_{n+1})<L(\tilde{v}_n)\}}\tilde{v}_n.$$

Notice that the sequence $L(\tilde{v}_n) = \sup_{i\leq n} L(v_i)$, so that $L(\tilde{v}_n)$ increases monotonically to $|L|_*$ and that $|\tilde{v}_n| \leq 1$ $\mathfrak{m}$-a.e. for every n. Also, for every $f \in L^\infty \cap L^2(\mathrm{X})$ we have $\|f\tilde{v}_n\| = \||f\tilde{v}_n|\|_{L^2} \leq \|f\|_{L^2}$ and thus we have

$$\int fL(v_n) \,\mathrm{d}\mathfrak{m} = \int L(fv_n) \,\mathrm{d}\mathfrak{m} \leq \|L\|_*\|f\tilde{v}_n\| \leq \|L\|_*\|f\|_{L^2}$$

so that letting $n \to \infty$ and using the monotone convergence theorem to pass to the limit in the left hand side, we obtain

$$\int f|L|_* \,\mathrm{d}\mathfrak{m} \leq \|L\|_*\|f\|_{L^2}$$

so that the arbitrariness of $f \in L^\infty \cap L^2(\mathrm{X})$ gives (16.3). □

Proposition 16.4 *Let $L : \mathscr{M} \to L^1(\mathrm{X})$ be linear, continuous and such that*

$$L(\chi_E v) = \chi_E L(v)$$

for every $v \in \mathscr{M}$ and $E \subset \mathrm{X}$ Borel. Then $L \in \mathscr{M}^*$.

Proof. We need to prove that

$$L(fv) = fL(v). \tag{16.4}$$

By assumption and taking into account the linearity of L we see that (16.4) is true for f simple. The claim then follows by the continuity of both sides of (16.4) in $f \in L^\infty$. $\square$

Exercise 16.5 *Assume that $\mathfrak{m}$ has no atoms and let $L : \mathscr{M} \to L^\infty(\mathrm{X})$ be linear, continuous and such that $L(fv) = fL(v)$ for every $v \in \mathscr{M}$ and $f \in L^\infty(\mathrm{X})$. Prove that $L \equiv 0$.*

We now study the relation between the dual module and the dual in the sense of Banach spaces. Thus let $\mathscr{M}'$ be the dual of $\mathscr{M}$ seen as Banach space. Integration provides a natural map $\mathrm{Int} : \mathscr{M}^* \to \mathscr{M}'$ sending $L \in \mathscr{M}^*$ to the operator $\mathrm{Int}(L) \in \mathscr{M}'$ defined as

$$\mathrm{Int}(L)(v) := \int L(v)\,\mathrm{d}\mathfrak{m}, \qquad \forall v \in \mathscr{M}.$$

Proposition 16.6 *The map* Int *is a bijective isometry, i.e. $\|L\|_* = \|\mathrm{Int}(L)\|_{\prime}$ for every $L \in \mathscr{M}^*$.*

Proof. From the inequality

$$|\mathrm{Int}(L)(v)| = \Big|\int L(v)\,\mathrm{d}\mathfrak{m}\Big| \le \|L(v)\|_{L^1(\mathrm{X})} \le \|v\|\|L\|_*$$

we see that $\|\mathrm{Int}(L)\|_{\prime} \le \|L\|_*$. For the converse inequality let $L \in \mathscr{M}^*$, fix $\varepsilon > 0$ and find $v \in \mathscr{M}$ such that $\|L(v)\|_{L^1} \ge \|v\|(\|L\|_* - \varepsilon)$. Put $\tilde v := \chi_{\{L(v)\ge 0\}} v - \chi_{\{L(v)<0\}} v$, notice that $|\tilde v| = |v|$ and $L(\tilde v) = |L(v)|$ $\mathfrak{m}$-a.e. and conclude by

$$\|\mathrm{Int}(L)\|_{\prime}\|\tilde v\| \ge |\mathrm{Int}(L)(\tilde v)| = \Big|\int L(\tilde v)\,\mathrm{d}\mathfrak{m}\Big| = \|L(v)\|_{L^1} \ge \|v\|(\|L\|_* - \varepsilon) = \|\tilde v\|(\|L\|_* - \varepsilon)$$

and the arbitrariness of $\varepsilon > 0$. It remains to prove that Int is surjective, hence fix $\ell \in \mathscr{M}'$ and for $v \in \mathscr{M}$ consider the map sending a Borel set $E \subset \mathrm{X}$ to $\mu_v(E) := \ell(\chi_E v) \in \mathbb{R}$. Clearly μ_v is additive and given a disjoint sequence (E_i) of Borel sets we have

$$|\mu_v(\cup_n E_n) - \mu_v(\cup_{n=1}^N E_n)| = |\mu_v(\cup_{n>N} E_n)| = |\ell(\chi_{\cup_{n>N} E_n} v)| \le \|\ell\|_{\prime}\|\chi_{\cup_{n>N} E_n} v\|$$

and since $\|\chi_{\cup_{n>N} E_n} v\|^2 = \int_{\cup_{n>N} E_n} |v|^2\,\mathrm{d}\mathfrak{m} \to 0$ by the dominated convergence theorem, we see that μ_v is a Borel measure. By construction, it is also absolutely continuous w.r.t. $\mathfrak{m}$ and thus it has a Radon-Nikodym derivative: call it $L(v) \in L^1(\mathrm{X})$.

By construction we have that $v \mapsto L(v)$ is linear. Also, since for every $E, F \subset \mathrm{X}$ Borel the identities $\mu_{\chi_E v}(F) = \ell(\chi_F \chi_E v) = \ell(\chi_{E\cap F} v) = \mu_v(E \cap F)$ grant that $\int_F L(\chi_E v) = \int_{E\cap F} L(v)$, we see that

$$L(\chi_E v) = \chi_E L(v) \qquad \forall v \in \mathscr{M},\ E \subset \mathrm{X} \text{ Borel}. \tag{16.5}$$

Now let us prove that $v \mapsto L(v) \in L^1(\mathrm{X})$ is continuous. For $v \in \mathscr{M}$ we put $\tilde{v} := \chi_{\{L(v)\geq 0\}}v - \chi_{\{L(v)<0\}}v$ so that $|\tilde{v}| = |v|$ and, by (16.5) and the linearity of L, we have $|L(v)| = L(\tilde{v})$ $\mathfrak{m}$-a.e.. Then

$$\|L(v)\|_{L^1} = \int L(\tilde{v})\,\mathrm{d}\mathfrak{m} = \mu_{\tilde{v}}(\mathrm{X}) = \ell(\tilde{v}) \leq \|\ell\|_{'}\|\tilde{v}\| = \|\ell\|_{'}\|v\|,$$

which was claim. The fact that $L \in \mathscr{M}^*$ now follows from (16.5) and Proposition 16.4. $\square$

17 Lesson [10/01/2018]

Let $\mathscr{M}$ be an $L^2(\mathfrak{m})$-normed module. Then the map

$$I_{\mathscr{M}} : \mathscr{M} \hookrightarrow \mathscr{M}^{**}, \quad \mathscr{M} \ni v \mapsto \Big(I_{\mathscr{M}}(v) : \mathscr{M}^* \ni L \mapsto L(v) \in L^1(\mathfrak{m})\Big) \in \mathscr{M}^{**} \tag{17.1}$$

is an isometric embedding. Indeed, its $L^\infty(\mathfrak{m})$-linearity can be easily proved, while to prove that it preserves the pointwise norm observe that

$$\big|I_{\mathscr{M}}(v)\big| = \operatorname*{ess\,sup}_{|L|_*\leq 1}\big|I_{\mathscr{M}}(v)(L)\big| = \operatorname*{ess\,sup}_{|L|_*\leq 1}\big|L(v)\big| \leq |v| \quad \mathfrak{m}\text{-a.e.} \quad \text{ for every } v \in \mathscr{M}$$

and that for any $v \in \mathscr{M}$ there exists $L \in \mathscr{M}^*$ such that $L(v) = |v|^2 = |L|_*^2$ holds $\mathfrak{m}$-a.e., namely choose $\ell \in \mathscr{M}'$ such that $\ell(v) = \|v\|^2_{\mathscr{M}} = \|\ell\|^2_{\mathscr{M}'}$ and set $L := \mathsf{Int}_{\mathscr{M}}^{-1}(\ell)$. Therefore one has that $\big|I_{\mathscr{M}}(v)\big| = |v|$ holds $\mathfrak{m}$-a.e. for every $v \in \mathscr{M}$, whence $I_{\mathscr{M}}$ is an isometric embedding.

Definition 17.1 *The $L^2(\mathfrak{m})$-normed module $\mathscr{M}$ is said to be* reflexive as module *provided the embedding $I_{\mathscr{M}}$ is surjective.*

Proposition 17.2 *The $L^2(\mathfrak{m})$-normed module $\mathscr{M}$ is reflexive as module if and only if it is reflexive as Banach space.*

Proof. The map $\mathsf{Int}_{\mathscr{M}} : \mathscr{M}^* \to \mathscr{M}'$ induces an isomorphism $\mathsf{Int}^{\mathrm{tr}}_{\mathscr{M}} : \mathscr{M}'' \to (\mathscr{M}^*)'$. Let us denote by $J : \mathscr{M} \hookrightarrow \mathscr{M}''$ the canonical embedding. We have that

$$\begin{aligned}\mathsf{Int}_{\mathscr{M}^*}\big(I_{\mathscr{M}}(v)\big)(L) &= \int I(v)(L)\,\mathrm{d}\mathfrak{m} = \int L(v)\,\mathrm{d}\mathfrak{m},\\ \mathsf{Int}^{\mathrm{tr}}_{\mathscr{M}}\big(J(v)\big)(L) &= J(v)\big(\mathsf{Int}_{\mathscr{M}}(L)\big) = \mathsf{Int}_{\mathscr{M}}(L)(v) = \int L(v)\,\mathrm{d}\mathfrak{m}\end{aligned}$$

for every $v \in \mathscr{M}$ and $L \in \mathscr{M}^*$, whence we deduce that the diagram

$$\begin{tikzcd} \mathscr{M} \arrow[r, hook, "I_{\mathscr{M}}"] \arrow[d, hook, "J"'] & \mathscr{M}^{**} \arrow[d, "\mathsf{Int}_{\mathscr{M}^*}"] \\ \mathscr{M}'' \arrow[r, "\mathsf{Int}^{\mathrm{tr}}_{\mathscr{M}}"'] & (\mathscr{M}^*)' \end{tikzcd}$$

commutes. Since $I_{\mathscr{M}}, J$ are injective and $\mathsf{Int}^{\mathrm{tr}}_{\mathscr{M}}$, $\mathsf{Int}_{\mathscr{M}^*}$ are bijective, we thus conclude that $I_{\mathscr{M}}$ is surjective if and only if J is surjective. $\square$

Proposition 17.3 *Let V be a generating linear subspace of $\mathscr{M}$. Suppose that $L : V \to L^1(\mathfrak{m})$ is a linear map such that for some $g \in L^2(\mathfrak{m})$ it holds*

$$\big|L(v)\big| \leq g\,|v| \quad \mathfrak{m}\text{-}a.e. \qquad \textit{for every } v \in V. \tag{17.2}$$

Then there exists a unique $\widetilde{L} \in \mathscr{M}^*$ such that $\widetilde{L}_{|V} = L$ Moreover, the inequality $|L|_* \leq g$ holds $\mathfrak{m}$-a.e. in X.

Proof. We claim that for any $v, w \in V$ and $E \subseteq \mathrm{X}$ Borel we have that

$$v = w \quad \mathfrak{m}\text{-a.e. on } E \quad \Longrightarrow \quad L(v) = L(w) \quad \mathfrak{m}\text{-a.e. on } E. \tag{17.3}$$

Indeed, note that (17.2) yields $\big|L(v) - L(w)\big| \leq \big|L(v-w)\big| \leq g\,|v-w| = 0$ $\mathfrak{m}$-a.e. on E. Now call $\widetilde{V}$ the set of all elements $\sum_{i=1}^n \chi_{E_i} v_i$, with $(E_i)_{i=1}^n$ Borel partition of X and $v_1, \ldots, v_n \in V$. The vector space $\widetilde{V}$ is dense in $\mathscr{M}$ by hypothesis. We are forced to define $\widetilde{L} : \widetilde{V} \to L^1(\mathfrak{m})$ as follows: $\widetilde{L}(\widetilde{v}) := \sum_{i=1}^n \chi_{E_i} L(v_i)$ for every $\widetilde{v} = \sum_{i=1}^n \chi_{E_i} v_i \in \widetilde{V}$, which is well-posed by (17.3) and linear by construction. Given that for every $\widetilde{v} = \sum_{i=1}^n \chi_{E_i} v_i \in \widetilde{V}$ we have

$$\big|\widetilde{L}(\widetilde{v})\big| = \sum_{i=1}^n \chi_{E_i} \big|L(v_i)\big| \leq g \sum_{i=1}^n \chi_{E_i} |v_i| = g\,|\widetilde{v}| \quad \mathfrak{m}\text{-a.e.}, \tag{17.4}$$

we deduce that $\big\|\widetilde{L}(\widetilde{v})\big\|_{L^1(\mathfrak{m})} \leq \|g\|_{L^2(\mathfrak{m})} \|\widetilde{v}\|_{\mathscr{M}}$ for every $\widetilde{v} \in \widetilde{V}$. In particular $\widetilde{L}$ is continuous, whence it can be uniquely extended to a linear and continuous map $\widetilde{L} : \mathscr{M} \to L^1(\mathfrak{m})$. It is easy to see that $\widetilde{L}$ is $L^\infty(\mathfrak{m})$-linear, so that $\widetilde{L} \in \mathscr{M}^*$. To conclude, the fact that the $\mathfrak{m}$-a.e. inequality $\big|\widetilde{L}(v)\big| \leq g\,|v|$ holds for every $v \in \mathscr{M}$ follows from (17.4) via an approximation argument. Hence $|L|_* \leq g$ holds $\mathfrak{m}$-a.e., as required. $\square$

Definition 17.4 (Tangent module) *We define the* tangent module $L^2(T\mathrm{X})$ *as the module dual of $L^2(T^*\mathrm{X})$. Its elements are called* vector fields*.*

We can introduce the notion of vector field in an alternative way, which is not based upon the theory of normed modules. Namely, we can define a suitable notion of derivation:

Definition 17.5 (L^2-derivations) *A linear map $L : \mathrm{S}^2(\mathrm{X}) \to L^1(\mathfrak{m})$ is an* L^2-derivation *provided there exists $\ell \in L^2(\mathfrak{m})$ such that*

$$\big|L(f)\big| \leq \ell\,|Df| \quad \mathfrak{m}\text{-}a.e. \qquad \textit{for every } f \in \mathrm{S}^2(\mathrm{X}). \tag{17.5}$$

The relation between vector fields and derivations is described in the following result:

Proposition 17.6 *Given any $X \in L^2(T\mathrm{X})$, the map $\mathrm{S}^2(\mathrm{X}) \ni f \mapsto \mathrm{d}f(X)$ is a derivation.*

Conversely, for any derivation $L : \mathrm{S}^2(\mathrm{X}) \to L^1(\mathfrak{m})$ there exists a unique $X \in L^2(T\mathrm{X})$ such that $L(f) = \mathrm{d}f(X)$.

Proof. Given any $X \in L^2(T\mathrm{X})$, let us define $L := X \circ \mathrm{d}$. Since $\big|L(f)\big| = \big|\mathrm{d}f(X)\big| \leq |Df||X|$ holds $\mathfrak{m}$-a.e., we have that L is the required derivation.

On the other hand, fix a derivation L and set $V := \big\{\mathrm{d}f \,:\, f \in \mathrm{S}^2(\mathrm{X})\big\}$. By arguing as in the proof of Proposition 17.3 one can see that for any $f_1, f_2 \in \mathrm{S}^2(\mathrm{X})$ we have

$$\mathrm{d}f_1 = \mathrm{d}f_2 \quad \mathfrak{m}\text{-a.e. on X} \qquad \Longrightarrow \qquad L(f_1) = L(f_2) \quad \mathfrak{m}\text{-a.e. on X}. \tag{17.6}$$

Then the map $T : V \to L^1(\mathfrak{m})$, given by $T(\mathrm{d}f) := L(f)$, is well-defined. Moreover, one has that $\big|T(\mathrm{d}f)\big| \leq \ell\,|Df|$ for each $f \in \mathrm{S}^2(\mathrm{X})$, whence Proposition 17.3 grants the existence of a unique vector field $X \in L^2(T\mathrm{X})$ such that $\omega(X) = T(\omega)$ for all $\omega \in V$. In other words, we have $\mathrm{d}f(X) = L(f)$ for every $f \in \mathrm{S}^2(\mathrm{X})$, getting the thesis. $\square$

Corollary 17.7 *Let $L : \mathrm{S}^2(\mathrm{X}) \to L^1(\mathfrak{m})$ be a derivation. Then*

$$L(fg) = f\,L(g) + g\,L(f) \qquad \textit{for every } f, g \in \mathrm{S}^2(\mathrm{X}) \cap L^\infty(\mathfrak{m}). \tag{17.7}$$

Proof. Direct consequence of Proposition 17.6 and of the Leibniz rule for the differential (see item C) of Theorem 14.1). $\square$

The adjoint $\mathrm{d}^* : L^2(T\mathrm{X}) \to L^2(\mathfrak{m})$ of the unbounded operator $\mathrm{d} : L^2(\mathfrak{m}) \to L^2(T^*\mathrm{X})$ is (up to a sign) what we call 'divergence operator'. More explicitly:

Definition 17.8 (Divergence) *We call $D(\mathrm{div})$ the space of all vector fields $X \in L^2(T\mathrm{X})$ for which there exists $h \in L^2(\mathfrak{m})$ satisfying*

$$-\int f\,h\,\mathrm{d}\mathfrak{m} = \int \mathrm{d}f(X)\,\mathrm{d}\mathfrak{m} \qquad \textit{for every } f \in W^{1,2}(\mathrm{X}). \tag{17.8}$$

The function h, which is unique by density of $W^{1,2}(\mathrm{X})$ in $L^2(\mathfrak{m})$, will be unambiguosly denoted by $\mathrm{div}(X)$. Moreover, $D(\mathrm{div})$ is a vector subspace of $L^2(T\mathrm{X})$ and $\mathrm{div} : D(\mathrm{div}) \to L^2(\mathfrak{m})$ is a linear operator.

We show some properties of the divergence operator:

Proposition 17.9 *Let $X, Y \in L^2(T\mathrm{X})$ be given. Suppose that $X = Y$ holds $\mathfrak{m}$-a.e. on some open set $\Omega \subseteq \mathrm{X}$. Then $\mathrm{div}(X) = \mathrm{div}(Y)$ is satisfied $\mathfrak{m}$-a.e. on Ω.*

Proof. By linearity of the divergence, it clearly suffices to prove that $\mathrm{div}(X) = 0$ $\mathfrak{m}$-a.e. on Ω whenever $X = 0$ $\mathfrak{m}$-a.e. on Ω. In order to prove it, notice that a simple cut-off argument gives

$$A := \big\{f \in W^{1,2}(\mathrm{X}) \,:\, f = 0 \text{ on } \Omega^c\big\} \text{ is dense in } B := \big\{g \in L^2(\mathfrak{m}) \,:\, g = 0 \text{ on } \Omega^c\big\}. \tag{17.9}$$

Moreover, $-\int f\,\mathrm{div}(X)\,\mathrm{d}\mathfrak{m} = \int \mathrm{d}f(X)\,\mathrm{d}\mathfrak{m} = 0$ holds for every $f \in A$, whence property (17.9) ensures that $\int g\,\mathrm{div}(X)\,\mathrm{d}\mathfrak{m} = 0$ for all $g \in B$, i.e. $\mathrm{div}(X)$ vanishes $\mathfrak{m}$-a.e. on Ω. $\square$

Proposition 17.10 *Let $X \in D(\mathrm{div})$ be given. Let $f : \mathrm{X} \to \mathbb{R}$ be any bounded Lipschitz function. Then $fX \in D(\mathrm{div})$ and*

$$\mathrm{div}(fX) = \mathrm{d}f(X) + f\,\mathrm{div}(X) \quad \textit{holds } \mathfrak{m}\textit{-a.e. in } \mathrm{X}. \tag{17.10}$$

Proof. Observe that the right hand side in (17.10) belongs to $L^2(\mathfrak{m})$. Then pick $g \in W^{1,2}(\mathrm{X})$. By the Leibniz rule for the differential, we have that

$$\begin{aligned} -\int g\big(\mathrm{d}f(X) + f\,\mathrm{div}(X)\big)\,\mathrm{d}\mathfrak{m} &= -\int g\,\mathrm{d}f(X) + fg\,\mathrm{div}(X)\,\mathrm{d}\mathfrak{m} = \int \mathrm{d}(fg)(X) - g\,\mathrm{d}f(X)\,\mathrm{d}\mathfrak{m} \\ &= \int f\,\mathrm{d}g(X)\,\mathrm{d}\mathfrak{m}. \end{aligned}$$

Therefore the thesis is achieved. □

We introduce a special class of vector fields: that of *gradients* of Sobolev functions.

Definition 17.11 *Let $f \in \mathrm{S}^2(\mathrm{X})$. Then we call $\mathsf{Grad}(f)$ the set of all $X \in L^2(T\mathrm{X})$ such that*

$$\mathrm{d}f(X) = |\mathrm{d}f|^2 = |X|^2 \quad \textit{holds } \mathfrak{m}\textit{-a.e. in } \mathrm{X}. \tag{17.11}$$

Remark 17.12 As seen above, it holds that $\mathsf{Grad}(f) \neq \emptyset$ for every $f \in \mathrm{S}^2(\mathrm{X})$. However, it can happen that $\mathsf{Grad}(f)$ is not a singleton. Furthermore, even if each $\mathsf{Grad}(f)$ is a singleton, its unique element does not necessarily depend linearly on f. ■

Given any Banach space $\mathbb{B}$, we can define the multi-valued map $\mathsf{Dual} : \mathbb{B} \twoheadrightarrow \mathbb{B}'$ as

$$\mathbb{B} \ni v \longmapsto \Big\{ L \in \mathbb{B}' \;:\; L(v) = \|L\|_{\mathbb{B}'}^2 = \|v\|_{\mathbb{B}}^2 \Big\}. \tag{17.12}$$

The Hahn-Banach theorem grants that $\mathsf{Dual}(v) \neq \emptyset$ for every $v \in \mathbb{B}$.

Exercise 17.13 Prove that Dual is single-valued and linear if and only if $\mathbb{B}$ is a Hilbert space. In this case, Dual is the Riesz isomorphism.

Coming back to the gradients, we point out that

$$\mathsf{Int}_{L^2(T^*\mathrm{X})}\big(\mathsf{Grad}(f)\big) = \mathsf{Dual}(\mathrm{d}f) \quad \text{for every } f \in \mathrm{S}^2(\mathrm{X}), \tag{17.13}$$

where the map Dual is associated to $\mathbb{B} := L^2(T^*\mathrm{X})$.

Example 17.14 Consider the space $(\mathbb{R}^2, \|\cdot\|_\infty)$, where $\big\|(x,y)\big\|_\infty = \max\big\{|x|,|y|\big\}$. Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ as $f(x,y) := x$. Then $\mathsf{Grad}(f) = \big\{(x,y) \in \mathbb{R}^2 \,:\, x = 1,\ |y| \leq 1\big\}$.

Exercise 17.15 Prove that Dual on $(\mathbb{R}^n, \|\cdot\|)$ is single-valued if and only if the norm $\|\cdot\|$ is of class C^1 (or, equivalently, the dual norm $\|\cdot\|_*$ is strictly convex).

Remark 17.16 The inequality $\mathrm{d}f(X) \leq \frac{1}{2}|\mathrm{d}f|^2 + \frac{1}{2}|X|^2$ holds $\mathfrak{m}$-a.e. in X for every $f \in \mathrm{S}^2(\mathrm{X})$ and $X \in L^2(T\mathrm{X})$ (by Young inequality). It can be readily proved that the opposite inequality is satisfied if and only if $X \in \mathsf{Grad}(f)$. ■

Theorem 17.17 *The following hold:*

A) Locality. *Let $f, g \in \mathrm{S}^2(\mathrm{X})$. Suppose that $f = g$ holds $\mathfrak{m}$-a.e. on some Borel set $E \subseteq \mathrm{X}$. Then for any $X \in \mathsf{Grad}(f)$ there exists $Y \in \mathsf{Grad}(g)$ such that $X = Y$ $\mathfrak{m}$-a.e. on E.*

B) Chain rule. *Let $f \in \mathrm{S}^2(\mathrm{X})$ and $X \in \mathsf{Grad}(f)$ be given.*

B1) *If a Borel set $N \subseteq \mathbb{R}$ is $\mathcal{L}^1$-negligible, then $X = 0$ holds $\mathfrak{m}$-a.e. on $f^{-1}(N)$.*

B2) *If $\varphi : \mathbb{R} \to \mathbb{R}$ is Lipschitz then $\varphi' \circ f\, \mathrm{X} \in \mathsf{Grad}(\varphi \circ f)$, where $\varphi' \circ f$ is arbitrarily defined on $f^{-1}\big\{$non-differentiability points of $\varphi\big\}$.*

Proof. To prove A), choose any $\widetilde{Y} \in \mathsf{Grad}(g)$ and define $Y := \chi_E\, X + \chi_{E^c}\, \widetilde{Y}$. Then $Y \in \mathsf{Grad}(g)$ and $X = Y$ $\mathfrak{m}$-a.e. on E, as required.

Property B1) directly follows from the analogous one for differentials (see Theorem 14.1), while to show B2) notice that

$$\begin{aligned} \mathrm{d}(\varphi \circ f)(\varphi' \circ f\, X) = \varphi' \circ f\, \mathrm{d}(\varphi \circ f)(X) = |\varphi' \circ f|^2\, \mathrm{d}f(X) = |\varphi' \circ f|^2\, |\mathrm{d}f|^2 = |\varphi' \circ f|^2\, |X|^2 \\ = \big|\mathrm{d}(\varphi \circ f)\big|^2 \end{aligned}$$

is verified $\mathfrak{m}$-a.e. on X. ☐

18 Lesson [15/01/2018]

In the previous lesson we used the following result, which we now fully justify:

Proposition 18.1 *Let $f \in W^{1,2}(\mathrm{X})$ and $g \in \mathrm{LIP}(\mathrm{X}) \cap L^\infty(\mathfrak{m})$ be given. Then $fg \in W^{1,2}(\mathrm{X})$ and $\mathrm{d}(fg) = f\, \mathrm{d}g + g\, \mathrm{d}f$.*

Proof. Fix $\bar{x} \in \mathrm{X}$ and for any $m \in \mathbb{N}$ pick a 1-Lipschitz function $\chi_m : \mathrm{X} \to [0,1]$ with bounded support such that $\chi_m = 1$ on $B_m(\bar{x})$. Then define $f_n := (f \wedge n) \vee (-n)$ and $g_m := \chi_m\, g$ for every $n, m \in \mathbb{N}$. Hence $f_n\, g_m \in W^{1,2}(\mathrm{X}) \cap L^\infty(\mathfrak{m})$ and $\mathrm{d}(f_n\, g_m) = f_n\, \mathrm{d}g_m + g_m\, \mathrm{d}f_n$. Given that $\big|\mathrm{d}(f_n\, g_m)\big| \leq \big(\|g\|_{L^\infty(\mathfrak{m})} + \mathrm{Lip}(g)\big)|f| + \|g\|_{L^\infty(\mathfrak{m})}|\mathrm{d}f| \in L^2(\mathfrak{m})$ holds $\mathfrak{m}$-a.e. for every choice of $n, m \in \mathbb{N}$ and $f_n\, g_m \to fg$ pointwise $\mathfrak{m}$-a.e. as $n, m \to \infty$, we deduce that $fg \in \mathrm{S}^2(\mathrm{X})$ by the closure of the differential. Now observe that for any $n \in \mathbb{N}$ we have

$$\chi_{B_m(\bar{x})}\, \mathrm{d}(f_n\, g) = \chi_{B_m(\bar{x})}\, \mathrm{d}(f_n\, g_m) = \chi_{B_m(\bar{x})}\big(f_n\, \mathrm{d}g + g\, \mathrm{d}f_n\big) \quad \text{for every } m \in \mathbb{N},$$

whence $\mathrm{d}(f_n\, g) = f_n\, \mathrm{d}g + g\, \mathrm{d}f_n$ is satisfied for every $n \in \mathbb{N}$. Given that $f_n\, g \to fg$ in $L^2(\mathfrak{m})$ and $f_n\, \mathrm{d}g + g\, \mathrm{d}f_n \to f\, \mathrm{d}g + g\, \mathrm{d}f$ in $L^2(T^*\mathrm{X})$, we conclude that $\mathrm{d}(fg) = f\, \mathrm{d}g + g\, \mathrm{d}f$ by the closure of d. ☐

Given any two Sobolev functions $f, g \in \mathrm{S}^2(\mathrm{X})$, let us define

$$H_{f,g}(\varepsilon) := \frac{1}{2}\big|D(g + \varepsilon\, f)\big|^2 \in L^1(\mathfrak{m}) \quad \text{for every } \varepsilon \in \mathbb{R}. \tag{18.1}$$

Then the map $H_{f,g} : \mathbb{R} \to L^1(\mathfrak{m})$ can be easily proven to be *convex*, meaning that

$$H\big((1-\lambda)\varepsilon_0+\lambda\varepsilon_1\big) \le (1-\lambda)H(\varepsilon_0)+\lambda H(\varepsilon_1) \quad \mathfrak{m}\text{-a.e.} \quad \text{for all } \varepsilon_0,\varepsilon_1 \in \mathbb{R} \text{ and } \lambda \in [0,1]. \tag{18.2}$$

Therefore the monotonicity of the incremental ratios of $H_{f,g}$ grants that

$$\exists\, L^1(\mathfrak{m})\text{-}\lim_{\varepsilon \searrow 0} \frac{H_{f,g}(\varepsilon) - H_{f,g}(0)}{\varepsilon} = \operatorname*{ess\,inf}_{\varepsilon>0} \frac{H_{f,g}(\varepsilon) - H_{f,g}(0)}{\varepsilon} \tag{18.3}$$

and an analogous statement holds for $\varepsilon \nearrow 0$.

Remark 18.2 The object in (18.3) could be morally denoted by $\mathrm{d}f(\nabla g)$, for the reasons we are now going to explain. Given a Banach space $\mathbb{B}$, we have that the map Dual defined in (17.12) is (formally) the differential of $\|\cdot\|^2_{\mathbb{B}}/2$. Since $T_v\mathbb{B} \approx \mathbb{B}$ and $T_{\|v\|^2_{\mathbb{B}}/2}\mathbb{R} \approx \mathbb{R}$ for any vector $v \in \mathbb{B}$, we can actually view $\mathrm{d}\big(\|\cdot\|^2_{\mathbb{B}}/2\big)(v) : T_v\mathbb{B} \to T_{\|v\|^2_{\mathbb{B}}/2}\mathbb{R}$ as an element of $\mathbb{B}'$. In our case, if we let $\mathbb{B} = L^2(T^*\mathrm{X})$ then we have that

$$\lim_{\varepsilon\to 0} \frac{\|\mathrm{d}g + \varepsilon\,\mathrm{d}f\|^2_{\mathbb{B}} - \|\mathrm{d}g\|^2_{\mathbb{B}}}{2\,\varepsilon} = \mathrm{d}\left(\frac{\|\cdot\|^2_{\mathbb{B}}}{2}\right)(\mathrm{d}g)(\mathrm{d}f) = \mathsf{Dual}(\mathrm{d}g)(\mathrm{d}f) = \mathrm{d}f(\nabla g),$$

which leads to our interpretation. ■

Proposition 18.3 *Let $f, g \in \mathrm{S}^2(\mathrm{X})$. Then the following hold:*

i) *for any $X \in \mathsf{Grad}(g)$ we have that $\operatorname{ess\,inf}_{\varepsilon>0} \frac{H_{f,g}(\varepsilon)-H_{f,g}(0)}{\varepsilon} \ge \mathrm{d}f(X)$ holds $\mathfrak{m}$-a.e. in X,*

ii) *there exists $X_{f,+} \in \mathsf{Grad}(g)$ such that $\operatorname{ess\,inf}_{\varepsilon>0} \frac{H_{f,g}(\varepsilon)-H_{f,g}(0)}{\varepsilon} = \mathrm{d}f(X_{f,+})$ $\mathfrak{m}$-a.e. in X,*

i′) *for any $X \in \mathsf{Grad}(g)$ we have that $\operatorname{ess\,sup}_{\varepsilon<0} \frac{H_{f,g}(\varepsilon)-H_{f,g}(0)}{\varepsilon} \le \mathrm{d}f(X)$ holds $\mathfrak{m}$-a.e. in X,*

ii′) *there exists $X_{f,-} \in \mathsf{Grad}(g)$ such that $\operatorname{ess\,sup}_{\varepsilon<0} \frac{H_{f,g}(\varepsilon)-H_{f,g}(0)}{\varepsilon} = \mathrm{d}f(X_{f,-})$ $\mathfrak{m}$-a.e. in X.*

Proof. i), i′) Take $X \in \mathsf{Grad}(g)$. By Remark 17.16 we have that

$$\mathrm{d}g(X) \ge \frac{1}{2}\,|\mathrm{d}g|^2 + \frac{1}{2}\,|X|^2 \quad \text{holds } \mathfrak{m}\text{-a.e. in } \mathrm{X}. \tag{18.4}$$

Moreover, an application of Young's inequality yields

$$\mathrm{d}(g+\varepsilon\,f)(X) \le \frac{1}{2}\,\big|\mathrm{d}(g+\varepsilon\,f)\big|^2 + \frac{1}{2}\,|X|^2 \quad \mathfrak{m}\text{-a.e. in } \mathrm{X}. \tag{18.5}$$

By subtracting (18.4) from (18.5) we thus obtain

$$\varepsilon\,\mathrm{d}f(X) \le \frac{\big|\mathrm{d}(g+\varepsilon\,f)\big|^2 - |\mathrm{d}g|^2}{2} \quad \mathfrak{m}\text{-a.e. in } \mathrm{X}. \tag{18.6}$$

Dividing both sides of (18.6) by $\varepsilon > 0$ (resp. $\varepsilon < 0$) and letting $\varepsilon \to 0$, we get i) (resp. i′)).
ii), ii′) We shall only prove ii), since the proof of ii′) is analogous. For any $\varepsilon \in (0,1)$, let us pick some $X_\varepsilon \in \mathsf{Grad}(g+\varepsilon\,f)$. Notice that

$$\|X_\varepsilon\|_{L^2(T\mathrm{X})} = \big\|\mathrm{d}(g+\varepsilon\,f)\big\|_{L^2(T^*\mathrm{X})} \le \|\mathrm{d}g\|_{L^2(T^*\mathrm{X})} + \|\mathrm{d}f\|_{L^2(T^*\mathrm{X})} \quad \text{for every } \varepsilon \in (0,1),$$

whence the intersection among all $0<\varepsilon'<1$ of the weak*-closure of $\big\{X_\varepsilon\,:\,\varepsilon\in(0,\varepsilon')\big\}$ is non-empty by Banach-Alaoglu theorem. Then call $X_{f,+}$ one of its elements. By expanding the formula $\mathrm{d}(g+\varepsilon\,f)(X_\varepsilon)\geq\frac{1}{2}\big|\mathrm{d}(g+\varepsilon\,f)\big|^2+\frac{1}{2}|X_\varepsilon|^2$, which holds $\mathfrak{m}$-a.e. for every $\varepsilon\in(0,1)$, we see that

$$\frac{1}{2}\,|X_\varepsilon|^2+\frac{1}{2}\,|\mathrm{d}g|^2-\mathrm{d}g(X_\varepsilon)\leq G_\varepsilon\quad\text{ holds }\mathfrak{m}\text{-a.e. in X,}\tag{18.7}$$

for a suitable $G_\varepsilon\in L^1(\mathfrak{m})$ that $L^1(\mathfrak{m})$-converges to 0 as $\varepsilon\searrow 0$. Observe that for any $E\subseteq\mathrm{X}$ Borel we have that

$$F_E:\,L^2(T\mathrm{X})\to\mathbb{R},\quad X\longmapsto\int_E\frac{1}{2}\,|X|^2+\frac{1}{2}\,|\mathrm{d}g|^2-\mathrm{d}g(X)\,\mathrm{d}\mathfrak{m}\tag{18.8}$$

is a weakly*-lower semicontinuous operator. Hence (18.7) grants that $F_E(X_{f,+})\leq 0$ for every Borel set $E\subseteq\mathrm{X}$, or equivalently $\frac{1}{2}\,|X_{f,+}|^2+\frac{1}{2}\,|\mathrm{d}g|^2-\mathrm{d}g(X_{f,+})\leq 0$ $\mathfrak{m}$-a.e. in X. Therefore Remark 17.16 gives $X_{f,+}\in\mathsf{Grad}(g)$. Finally, observe that it $\mathfrak{m}$-a.e. holds

$$\mathrm{d}f(X_\varepsilon)\geq\operatorname*{ess\,inf}_{\varepsilon'>0}\frac{H_{f,g}(\varepsilon')-H_{f,g}(0)}{\varepsilon'}=:\Theta\quad\text{ for every }\varepsilon\in(0,1).\tag{18.9}$$

Recall that $L^2(T\mathrm{X})\ni X\mapsto\int\omega(X)\,\mathrm{d}\mathfrak{m}$ is weakly*-continuous for any $\omega\in L^2(T^*\mathrm{X})$. By applying this fact with $\omega:=\chi_E\,\mathrm{d}f$, where $E\subseteq\mathrm{X}$ is any Borel set, we deduce from (18.9) that

$$\int_E\mathrm{d}f(X_{f,+})\,\mathrm{d}\mathfrak{m}\geq\int_E\Theta\,\mathrm{d}\mathfrak{m}\quad\text{ for every }E\subseteq\mathrm{X}\text{ Borel.}$$

This grants that $\mathrm{d}f(X_{f,+})\geq\Theta$ holds $\mathfrak{m}$-a.e. in X, which together with i) imply ii). □

Exercise 18.4 Prove that the square of the norm of a finite-dimensional Banach space is differentiable if and only if its dual norm is strictly convex.

Corollary 18.5 *The following are equivalent:*

i) *for every $f,g\in\mathrm{S}^2(\mathrm{X})$ it holds that*

$$\operatorname*{ess\,inf}_{\varepsilon>0}\frac{H_{f,g}(\varepsilon)-H_{f,g}(0)}{\varepsilon}=\operatorname*{ess\,sup}_{\varepsilon<0}\frac{H_{f,g}(\varepsilon)-H_{f,g}(0)}{\varepsilon},\tag{18.10}$$

ii) *for every $g\in\mathrm{S}^2(\mathrm{X})$ the set $\mathsf{Grad}(g)$ is a singleton.*

Proof. ii) $\Longrightarrow$ i) It trivially follows from items ii) and ii′) of Proposition 18.3.
i) $\Longrightarrow$ ii) Our aim is to show that if $X,Y\in\mathsf{Grad}(g)$ then $X=Y$. We claim that it is enough to prove that

$$\mathrm{d}f(X)=\mathrm{d}f(Y)\quad\text{ for every }f\in\mathrm{S}^2(\mathrm{X}).\tag{18.11}$$

Indeed, if (18.11) holds true then the operator $\mathrm{d}f\mapsto\mathrm{d}f(X-Y)$ from the generating linear subspace $V:=\big\{\mathrm{d}f\,:\,f\in\mathrm{S}^2(\mathrm{X})\big\}$ of $L^2(T^*\mathrm{X})$ to $L^1(\mathfrak{m})$ is identically null, whence accordingly we have that $X-Y=0$ by Proposition 17.3. This shows that it suffices to prove (18.11).

Take any $f \in \mathrm{S}^2(\mathrm{X})$. Suppose that (18.11) fails, then (possibly interchanging X and Y) there exists a Borel set $E \subseteq \mathrm{X}$ with $\mathfrak{m}(E) > 0$ such that $\mathrm{d}f(X) < \mathrm{d}f(Y)$ holds $\mathfrak{m}$-a.e. in E. Therefore we have that

$$\operatorname*{ess\,sup}_{\varepsilon<0} \frac{H_{f,g}(\varepsilon) - H_{f,g}(0)}{\varepsilon} \leq \mathrm{d}f(X) < \mathrm{d}f(Y) \leq \operatorname*{ess\,inf}_{\varepsilon>0} \frac{H_{f,g}(\varepsilon) - H_{f,g}(0)}{\varepsilon} \quad \mathfrak{m}\text{-a.e. in } E,$$

which contradicts (18.10). This shows (18.11), as required. □

Definition 18.6 (Infinitesimal strict convexity) *We say that* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is* infinitesimally strictly convex *provided the two conditions of Corollary 18.5 hold true. For any* $g \in \mathrm{S}^2(\mathrm{X})$*, we shall denote by* ∇g *the only element of* $\mathsf{Grad}(g)$*.*

Definition 18.7 (Hilbert module) *An* $L^2(\mathfrak{m})$*-normed module* $\mathscr{H}$ *is said to be a* Hilbert module *provided* $(\mathscr{H}, \|\cdot\|_{\mathscr{H}})$ *is a Hilbert space.*

Proposition 18.8 *Every Hilbert module is reflexive.*

Proof. Any Hilbert module is clearly reflexive when viewed as a Banach space, thus also in the sense of modules by Proposition 17.2. □

Proposition 18.9 *Let* $\mathscr{H}$ *be a Hilbert module. Then the formula*

$$\langle v, w \rangle := \frac{1}{2}\big(|v+w|^2 - |v|^2 - |w|^2\big) \in L^1(\mathfrak{m}) \tag{18.12}$$

defines an $L^\infty(\mathfrak{m})$*-bilinear map* $\langle \cdot, \cdot \rangle : \mathscr{H} \times \mathscr{H} \to L^1(\mathfrak{m})$*, called* pointwise scalar product*, which satisfies*

$$\left.\begin{aligned} \langle v, w \rangle &= \langle w, v \rangle \\ \big|\langle v, w \rangle\big| &\leq |v||w| \quad \mathfrak{m}\text{-a.e.} \\ \langle v, v \rangle &= |v|^2 \end{aligned}\right. \qquad \text{for every } v, w \in \mathscr{H}. \tag{18.13}$$

Moreover, the pointwise parallelogram rule *is satisfied, i.e.*

$$2\big(|v|^2 + |w|^2\big) = |v+w|^2 + |v-w|^2 \quad \mathfrak{m}\text{-a.e.} \qquad \text{for every } v, w \in \mathscr{H}. \tag{18.14}$$

Proof. We only prove formula (18.14). The other properties can be obtained by suitably adapting the proof of the analogous statements for Hilbert spaces, apart from the $L^\infty(\mathfrak{m})$-bilinearity of $\langle \cdot, \cdot \rangle$, which can be shown by using the fact that $\langle \cdot, \cdot \rangle$ is local and continuous with respect to both entries by its very construction. Then let $v, w \in \mathscr{H}$ be fixed. Since the norm $\|\cdot\|_{\mathscr{H}}$ satisfies the parallelogram rule, we have that for any Borel set $E \subseteq \mathrm{X}$ it holds

$$\begin{aligned} 2\int_E |v|^2 + |w|^2 \,\mathrm{d}\mathfrak{m} &= 2\,\|\chi_E\, v\|^2_{\mathscr{H}} + 2\,\|\chi_E\, w\|^2_{\mathscr{H}} = \|\chi_E\, v + \chi_E\, w\|^2_{\mathscr{H}} + \|\chi_E\, v - \chi_E\, w\|^2_{\mathscr{H}} \\ &= \int_E |v+w|^2 + |v-w|^2 \,\mathrm{d}\mathfrak{m}, \end{aligned}$$

which yields (18.14) by arbitrariness of E. □

Given any Hilbert module $\mathscr{H}$, it holds that

$$\int \langle v, w \rangle \, \mathrm{d}\mathfrak{m} = \langle v, w \rangle_{\mathscr{H}} \quad \text{for every } v, w \in \mathscr{H}, \tag{18.15}$$

as one can immediately see by recalling that $\int |v|^2 \, \mathrm{d}\mathfrak{m} = \|v\|^2_{\mathscr{H}}$.

Remark 18.10 Actually the pointwise parallelogram rule characterises the Hilbert modules: any $L^2(\mathfrak{m})$-normed module is a Hilbert module if and only if (18.14) is satisfied. ■

Theorem 18.11 (Riesz) *Let $\mathscr{H}$ be a Hilbert module. Then for every $L \in \mathscr{H}^*$ there exists a unique element $v \in \mathscr{H}$ such that*

$$L(w) = \langle v, w \rangle \quad \textit{for every } w \in \mathscr{H}. \tag{18.16}$$

Moreover, the equality $|v| = |L|_*$ holds $\mathfrak{m}$-a.e. in X.

Proof. Consider $\mathsf{Int}(L) \in \mathscr{H}'$. By the classical Riesz theorem, there exists (a unique) $v \in \mathscr{H}$ such that $\langle v, w \rangle_{\mathscr{H}} = \mathsf{Int}(L)(w)$ for every $w \in \mathscr{H}$. Hence for any $w \in \mathscr{H}$ we have that

$$\int_E \langle v, w \rangle \, \mathrm{d}\mathfrak{m} = \langle v, \chi_E \, w \rangle_{\mathscr{H}} = \mathsf{Int}(L)(\chi_E \, w) = \int_E L(w) \, \mathrm{d}\mathfrak{m} \quad \text{for every } E \subseteq \mathrm{X} \text{ Borel},$$

so that (18.16) is satisfied. Finally, it is easy to show that $|v| = \operatorname{ess\,sup}_{|w| \leq 1} \langle v, w \rangle$. Recall that also $|L|_* = \operatorname{ess\,sup}_{|w| \leq 1} L(w)$, therefore the $\mathfrak{m}$-a.e. equality $|v| = |L|_*$ follows. □

It immediately follows from Theorem 18.11 that the map $\mathscr{H} \ni v \mapsto \langle v, \cdot \rangle \in \mathscr{H}^*$ is an isometric isomorphism of modules.

Example 18.12 We compare the Riesz theorem for Hilbert spaces and Theorem 18.11 in the special case in which $\mathscr{H} = L^2(\mathfrak{m})$.

The former grants that for any linear and continuous map $\ell : L^2(\mathfrak{m}) \to \mathbb{R}$ there exists a unique g in $L^2(\mathfrak{m})$ such that $\ell(f) = \int f g \, \mathrm{d}\mathfrak{m}$ for every $f \in L^2(\mathfrak{m})$, thus $\|g\|_{L^2(\mathfrak{m})} = \|\ell\|_{L^2(\mathfrak{m})'}$.

The latter grants that for any $L^\infty(\mathfrak{m})$-linear and continuous map $L : L^2(\mathfrak{m}) \to L^1(\mathfrak{m})$ there exists a unique g in $L^2(\mathfrak{m})$ such that $L(f) = f g$ holds $\mathfrak{m}$-a.e. for every $f \in L^2(\mathfrak{m})$, thus accordingly $|g| = |L|_*$ holds $\mathfrak{m}$-a.e. in X.

19 Lesson [17/01/2018]

Theorem 19.1 *The following are equivalent:*

i) $W^{1,2}(\mathrm{X})$ *is a Hilbert space,*

ii) $2\big(|\mathrm{d}f|^2 + |\mathrm{d}g|^2\big) = \big|\mathrm{d}(f+g)\big|^2 + \big|\mathrm{d}(f-g)\big|^2$ *holds $\mathfrak{m}$-a.e. for every $f, g \in W^{1,2}(\mathrm{X})$,*

iii) $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is infinitesimally strictly convex and $\mathrm{d}f(\nabla g) = \mathrm{d}g(\nabla f)$ holds $\mathfrak{m}$-a.e. in* X *for every $f, g \in W^{1,2}(\mathrm{X})$,*

iv) $L^2(T^*\mathrm{X})$ *and* $L^2(T\mathrm{X})$ *are Hilbert modules,*

v) $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *is infinitesimally strictly convex and* $\nabla(f+g)=\nabla f+\nabla g$ *holds* $\mathfrak{m}$*-a.e. in* X *for every* $f,g\in W^{1,2}(\mathrm{X})$*,*

vi) $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *is infinitesimally strictly convex and* $\nabla(fg)=f\,\nabla g+g\,\nabla f$ *holds* $\mathfrak{m}$*-a.e. in* X *for every* $f,g\in W^{1,2}(\mathrm{X})\cap L^\infty(\mathfrak{m})$*.*

Proof. i) $\Longrightarrow$ ii) First of all, observe that $W^{1,2}(\mathrm{X})$ is a Hilbert space if and only if

$$W^{1,2}(\mathrm{X})\ni f\longmapsto \mathsf{E}(f):=\frac{1}{2}\int|\mathrm{d}f|^2\,\mathrm{d}\mathfrak{m}\quad\text{ satisfies the parallelogram rule.}\tag{19.1}$$

Now suppose that i) holds, then $\mathsf{E}(f+\varepsilon\,g)+\mathsf{E}(f-\varepsilon\,g)=2\,\mathsf{E}(f)+2\,\varepsilon^2\,\mathsf{E}(g)$ for all $f,g\in W^{1,2}(\mathrm{X})$ and $\varepsilon\neq 0$, or equivalently

$$\frac{\mathsf{E}(f+\varepsilon\,g)-\mathsf{E}(f)}{\varepsilon}-\frac{\mathsf{E}(f-\varepsilon\,g)-\mathsf{E}(f)}{\varepsilon}=2\,\varepsilon\,\mathsf{E}(g).\tag{19.2}$$

Hence (19.2) and Proposition 18.3 grant that

$$\begin{aligned}\int\operatorname*{ess\,sup}_{X\in\mathsf{Grad}(f)}\mathrm{d}g(X)\,\mathrm{d}\mathfrak{m}&=\lim_{\varepsilon\searrow 0}\frac{\mathsf{E}(f+\varepsilon\,g)-\mathsf{E}(f)}{\varepsilon}=\lim_{\varepsilon\nearrow 0}\frac{\mathsf{E}(f+\varepsilon\,g)-\mathsf{E}(f)}{\varepsilon}\\&=\int\operatorname*{ess\,inf}_{X\in\mathsf{Grad}(f)}\mathrm{d}g(X)\,\mathrm{d}\mathfrak{m},\end{aligned}$$

thus accordingly $\operatorname{ess\,inf}_{X\in\mathsf{Grad}(f)}\mathrm{d}g(X)=\operatorname{ess\,sup}_{X\in\mathsf{Grad}(f)}\mathrm{d}g(X)$ holds $\mathfrak{m}$-a.e. in X. This guarantees that $\mathsf{Grad}(f)$ is a singleton for every $f\in W^{1,2}(\mathrm{X})$, i.e. $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is infinitesimally strictly convex. We now claim that

$$\int\mathrm{d}f(\nabla g)\,\mathrm{d}\mathfrak{m}=\int\mathrm{d}g(\nabla f)\,\mathrm{d}\mathfrak{m}\quad\text{ for every }f,g\in W^{1,2}(\mathrm{X}).\tag{19.3}$$

Given $f,g\in W^{1,2}(\mathrm{X})$, denote by $\mathcal{Q}:\mathbb{R}^2\to\mathbb{R}$ the function $(t,s)\mapsto\mathsf{E}(t\,f+s\,g)$. Since $\mathcal{Q}$ is a quadratic polynomial, in particular smooth, we have $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\mathcal{Q}(t,s)=\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\mathcal{Q}(t,s)$. The left-hand side of the previous equation can be rewritten as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\left(\lim_{h\to 0}\frac{\mathsf{E}(t\,f+h\,g)-\mathsf{E}(t\,f)}{h}\right)&=\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\left(\int\mathrm{d}g\big(\nabla(t\,f)\big)\,\mathrm{d}\mathfrak{m}\right)\\&=\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\left(t\int\mathrm{d}g(\nabla f)\,\mathrm{d}\mathfrak{m}\right)=\int\mathrm{d}g(\nabla f)\,\mathrm{d}\mathfrak{m}\end{aligned}$$

and analogously the right-hand side equals $\int\mathrm{d}f(\nabla g)\,\mathrm{d}\mathfrak{m}$, proving (19.3).

Fix any function $h\in\mathrm{LIP}(\mathrm{X})\cap L^\infty(\mathfrak{m})$. We want to prove that

$$W^{1,2}(\mathrm{X})\cap L^\infty(\mathfrak{m})\ni f\longmapsto\int h\,|\mathrm{d}f|^2\,\mathrm{d}\mathfrak{m}\quad\text{ satisfies the parallelogram rule.}\tag{19.4}$$

To this aim, notice that the Leibniz rule and the chain rule for differentials yield

$$\begin{aligned}\int h\,|\mathrm{d}f|^2\,\mathrm{d}\mathfrak{m}&=\int h\,\mathrm{d}f(\nabla f)\,\mathrm{d}\mathfrak{m}=\int\mathrm{d}(fh)(\nabla f)-f\,\mathrm{d}h(\nabla f)\,\mathrm{d}\mathfrak{m}\\&=\int\mathrm{d}(fh)(\nabla f)-\mathrm{d}h\big(\nabla(f^2/2)\big)\,\mathrm{d}\mathfrak{m}\overset{(19.3)}{=}\int\mathrm{d}(fh)(\nabla f)-\mathrm{d}(f^2/2)(\nabla h)\,\mathrm{d}\mathfrak{m}.\end{aligned}$$

Both the addenda $\int \mathrm{d}(fh)(\nabla f)\,\mathrm{d}\mathfrak{m}$ and $-\int \mathrm{d}(f^2/2)(\nabla h)\,\mathrm{d}\mathfrak{m}$ are quadratic forms, the former because $(f,g) \mapsto \int \mathrm{d}(fh)(\nabla g)\,\mathrm{d}\mathfrak{m} = \int \mathrm{d}g\big(\nabla(fh)\big)\,\mathrm{d}\mathfrak{m}$ is bilinear, whence (19.4). Given that the set $\mathrm{LIP}(\mathrm{X}) \cap L^\infty(\mathfrak{m})$ is weakly* dense in $L^\infty(\mathfrak{m})$, we finally deduce from (19.4) that

$$2\int h\,|\mathrm{d}f|^2 + h\,|\mathrm{d}g|^2\,\mathrm{d}\mathfrak{m} = \int h\,\big|\mathrm{d}(f+g)\big|^2 + h\,\big|\mathrm{d}(f-g)\big|^2\,\mathrm{d}\mathfrak{m}$$

holds for every $f,g \in W^{1,2}(\mathrm{X})$ and $h \in L^\infty(\mathfrak{m})$. Therefore ii) follows.

ii) $\Longrightarrow$ i) By integrating the pointwise parallelogram rule over X, we get the parallelogram rule for $\|\cdot\|_{W^{1,2}(\mathrm{X})}$, so that $W^{1,2}(\mathrm{X})$ is a Hilbert space.

i) $\Longrightarrow$ iii) By arguing exactly as in the first implication, we see that $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is infinitesimally strictly convex and that (19.4) holds true. By following the argument we used to prove (19.3), we deduce that

$$\int h\,\mathrm{d}f(\nabla g)\,\mathrm{d}\mathfrak{m} = \int h\,\mathrm{d}g(\nabla f)\,\mathrm{d}\mathfrak{m} \qquad \begin{array}{l}\text{for every } f,g \in W^{1,2}(\mathrm{X}) \cap L^\infty(\mathfrak{m})\\ \text{and } h \in \mathrm{LIP}(\mathrm{X}) \cap L^\infty(\mathfrak{m}).\end{array} \tag{19.5}$$

Given that the set $\mathrm{LIP}(\mathrm{X}) \cap L^\infty(\mathfrak{m})$ is weakly* dense in $L^\infty(\mathfrak{m})$, we conclude from (19.5) (by applying a truncation and localisation argument) that $\mathrm{d}f(\nabla g) = \mathrm{d}g(\nabla f)$ holds $\mathfrak{m}$-a.e. for every $f,g \in W^{1,2}(\mathrm{X})$. This shows that iii) is verified.

iii) $\Longrightarrow$ i) It suffices to prove that E satisfies the parallelogram rule. Fix $f,g \in W^{1,2}(\mathrm{X})$. Note that the function $[0,1] \ni t \mapsto \mathsf{E}(f + t\,g)$ is Lipschitz and that its derivative is given by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\,\mathsf{E}(f+t\,g) &= \lim_{h\to 0}\frac{\mathsf{E}\big((f+t\,g)+h\,g\big) - \mathsf{E}(f+t\,g)}{h} = \int \mathrm{d}g\big(\nabla(f+t\,g)\big)\,\mathrm{d}\mathfrak{m}\\ &= \int \mathrm{d}(f+t\,g)(\nabla g)\,\mathrm{d}\mathfrak{m} = \int \mathrm{d}f(\nabla g)\,\mathrm{d}\mathfrak{m} + t\int |\mathrm{d}g|^2\,\mathrm{d}\mathfrak{m},\end{aligned}$$

whence by integrating on $[0,1]$ we get $\mathsf{E}(f+g) - \mathsf{E}(f) = \int \mathrm{d}f(\nabla g)\,\mathrm{d}\mathfrak{m} + \int |\mathrm{d}g|^2/2\,\mathrm{d}\mathfrak{m}$. If we replace g with $-g$, we also obtain that $\mathsf{E}(f-g) - \mathsf{E}(f) = -\int \mathrm{d}f(\nabla g)\,\mathrm{d}\mathfrak{m} + \int |\mathrm{d}g|^2/2\,\mathrm{d}\mathfrak{m}$, whence by summing these two equalities we conclude that $\mathsf{E}(f+g)+\mathsf{E}(f-g) = 2\,\mathsf{E}(f)+2\,\mathsf{E}(g)$.

ii) $\Longrightarrow$ iv) Consider two 1-forms ω and η in $L^2(T^*\mathrm{X})$, say $\omega = \sum_i \chi_{E_i}\mathrm{d}f_i$ and $\eta = \sum_j \chi_{F_j}\mathrm{d}g_j$. By locality we see that $|\omega+\eta|^2 + |\omega-\eta|^2 = 2\,|\omega|^2 + 2\,|\eta|^2$ holds $\mathfrak{m}$-a.e. in X, whence by integrating we get $\|\omega+\eta\|^2_{L^2(T^*\mathrm{X})} + \|\omega-\eta\|^2_{L^2(T^*\mathrm{X})} = 2\,\|\omega\|^2_{L^2(T^*\mathrm{X})} + 2\,\|\eta\|^2_{L^2(T^*\mathrm{X})}$. By density of the simple 1-forms in $L^2(T^*\mathrm{X})$, we conclude that $L^2(T^*\mathrm{X})$ (and accordingly also $L^2(T\mathrm{X})$) is a Hilbert module, thus proving iv).

iv) $\Longrightarrow$ ii) It trivially follows from Proposition 18.9.

iv) $\Longrightarrow$ v) Let $f \in W^{1,2}(\mathrm{X})$ and $X \in \mathsf{Grad}(f)$. By Theorem 18.11 applied to $L^2(T\mathrm{X})$ there exists a unique 1-form $\omega \in L^2(T^*\mathrm{X})$ such that $\langle\omega,\eta\rangle = \eta(X)$ for every $\eta \in L^2(T^*\mathrm{X})$. Moreover, it holds that $|\omega|_* = |X| = |\mathrm{d}f|_*$ $\mathfrak{m}$-a.e. in X. Hence by taking $\eta := \mathrm{d}f$ we see that

$$|\omega - \mathrm{d}f|_*^2 = |\omega|_*^2 + |\mathrm{d}f|_*^2 - 2\,\langle\omega,\mathrm{d}f\rangle = 2\,|\mathrm{d}f|_*^2 - \mathrm{d}f(X) = 0 \quad \mathfrak{m}\text{-a.e.},$$

which grants that $\omega = \mathrm{d}f$. Again by Theorem 18.11, we deduce that $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is infinitesimally strictly convex and that $f \mapsto \nabla f$ is linear, as required.

v) $\Longrightarrow$ ii) For any $f, g \in W^{1,2}(\mathrm{X})$, it $\mathfrak{m}$-a.e. holds that

$$\begin{aligned}
\left|\mathrm{d}(f+g)\right|^2 &= \mathrm{d}(f+g)\big(\nabla(f+g)\big) = \mathrm{d}f(\nabla f) + \mathrm{d}f(\nabla g) + \mathrm{d}g(\nabla f) + \mathrm{d}g(\nabla g),\\
\left|\mathrm{d}(f-g)\right|^2 &= \mathrm{d}(f-g)\big(\nabla(f-g)\big) = \mathrm{d}f(\nabla f) - \mathrm{d}f(\nabla g) - \mathrm{d}g(\nabla f) + \mathrm{d}g(\nabla g),
\end{aligned}$$

hence by summing them we get the $\mathfrak{m}$-a.e. equality $\left|\mathrm{d}(f+g)\right|^2 + \left|\mathrm{d}(f-g)\right|^2 = 2\,|\mathrm{d}f|^2 + 2\,|\mathrm{d}g|^2$, proving the validity of ii).

v) $\Longleftrightarrow$ vi) By applying the chain rule for gradients, we see that if $f, g \in W^{1,2}(\mathrm{X}) \cap L^\infty(\mathfrak{m})$ and $f' := \exp(f)$, $g' := \exp(g)$, then we have

$$\begin{aligned}
f'g'\,\nabla(f+g) &= f'g'\,\nabla\big(\log(f'g')\big) = \nabla(f'g'),\\
f'g'\big(\nabla f + \nabla g\big) &= f'g'\,\nabla\big(\log(f')\big) + f'g'\,\nabla\big(\log(g')\big) = g'\,\nabla f' + f'\,\nabla g'.
\end{aligned}$$

Therefore we conclude that v) is equivalent to vi), thus concluding the proof. □

Definition 19.2 (Infinitesimal Hilbertianity) *We say that* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is* infinitesimally Hilbertian *provided the six conditions of Theorem 19.1 hold true.*

Lemma 19.3 *Let $\mathscr{M}$ be an $L^2(\mathfrak{m})$-normed module. Let $S \subseteq \mathscr{M}$ be a separable subset with the following property: the $L^\infty(\mathfrak{m})$-linear combinations of elements of S are dense in $\mathscr{M}$. Then the space $\mathscr{M}$ is separable.*

Proof. Pick a countable dense subset $(v_n)_n$ of S. It is then clear that the $L^\infty(\mathfrak{m})$-linear combinations of the v_n's are dense in $\mathscr{M}$. It only remains to show that the family of such combinations is separable. Now fix a Borel probability measure $\mathfrak{m}'$ on X with $\mathfrak{m} \ll \mathfrak{m}' \ll \mathfrak{m}$. Then there exists a countable family $\mathcal{A}$ of Borel subsets of X such that for any $E \subseteq \mathrm{X}$ Borel there is a sequence $(E_i)_i \subseteq \mathcal{A}$ with $\mathfrak{m}'(E_i \Delta E) \to 0$. For instance, define $\mathcal{A}$ as the set of all open balls with rational radii that are centered at some fixed countable dense subset of X. Hence let us define the separable set D as

$$D := \left\{ \sum_{n=0}^{N} \alpha_n\,\chi_{E_n} v_n \;\middle|\; N \in \mathbb{N},\ (\alpha_n)_{n=0}^N \subseteq \mathbb{Q},\ (E_n)_{n=0}^N \subseteq \mathcal{A} \right\}.$$

It can be readily proved that the set of all $L^\infty(\mathfrak{m})$-linear combinations of the v_n's is contained in the closure of D. Therefore the thesis is achieved. □

Proposition 19.4 *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an infinitesimally Hilbertian metric measure space. Then the spaces $W^{1,2}(\mathrm{X})$, $L^2(T^*\mathrm{X})$ and $L^2(T\mathrm{X})$ are separable.*

Proof. The space $W^{1,2}(\mathrm{X})$, being reflexive by hypothesis, is separable by Theorem 9.7. Given that the differentials of the functions in $W^{1,2}(\mathrm{X})$ generate the cotangent module, we deduce from Lemma 19.3 that even $L^2(T^*\mathrm{X})$ is separable. Finally, Theorem 18.11 grants that $L^2(T\mathrm{X})$ is separable as well. □

We now introduce the notion of 'pullback module'. In order to explain the ideas underlying its construction, we first show how things work in the classical case of smooth manifolds.

Let $\varphi : M \to N$ be a smooth map between two smooth manifolds M and N. Given a point $x \in M$ and a tangent vector $v \in T_x M$, we have that $\mathrm{d}\varphi_x(v) \in T_{\varphi(x)}N$ is the unique element for which $\mathrm{d}\varphi_x(v)(f) = \mathrm{d}(f \circ \varphi)_x(v)$ holds for any smooth function f on N. However, in our framework vector fields are not pointwise defined, so we are rather interested in giving a meaning to the object $\mathrm{d}\varphi(X)$, where X is a vector field on M. Unless φ is a diffeomorphism, we cannot hope to define $\mathrm{d}\varphi(X)$ as a vector field on N. What we need is the notion of 'pullback bundle': informally speaking, given a bundle E over N, we define $\varphi^* E$ as that bundle over M such that the fiber at a point $x \in M$ is exactly the fiber of E at $\varphi(x)$. Hence the object $\mathrm{d}\varphi(X)$ can be defined as the section of $\varphi^* TN$ satisfying $\mathrm{d}\varphi(X)(x) = \mathrm{d}\varphi_x\big(X(x)\big)$ for every $x \in M$.

Definition 19.5 (Maps of bounded compression) *Let* $(\mathrm{X}, \mathsf{d}_\mathrm{X}, \mathfrak{m}_\mathrm{X})$ *and* $(\mathrm{Y}, \mathsf{d}_\mathrm{Y}, \mathfrak{m}_\mathrm{Y})$ *be metric measure spaces. Then a map* $\varphi : \mathrm{Y} \to \mathrm{X}$ *is said to be* of bounded compression *provided it is Borel and there exists a constant* $C > 0$ *such that* $\varphi_* \mathfrak{m}_\mathrm{Y} \leq C \mathfrak{m}_\mathrm{X}$. *The least such constant* $C > 0$ *will be denoted by* $\mathrm{Comp}(\varphi)$ *and called* compression constant *of* φ.

20 Lesson [22/01/2018]

We introduce the notion of 'pullback module'. The proof of the following result is only sketched, as it similar in spirit to that of Theorem 13.2.

Theorem 20.1 (Pullback module) *Let* $(\mathrm{X}, \mathsf{d}_\mathrm{X}, \mathfrak{m}_\mathrm{X})$ *and* $(\mathrm{Y}, \mathsf{d}_\mathrm{Y}, \mathfrak{m}_\mathrm{Y})$ *be metric measure spaces. Let* $\mathscr{M}$ *be an* $L^2(\mathfrak{m}_\mathrm{X})$*-normed module. Let* $\varphi : \mathrm{Y} \to \mathrm{X}$ *be a map of bounded compression. Then there exists a unique couple* $(\varphi^*\mathscr{M}, \varphi^*)$*, where* $\varphi^*\mathscr{M}$ *is an* $L^2(\mathfrak{m}_\mathrm{Y})$*-normed module and* $\varphi^* : \mathscr{M} \to \varphi^*\mathscr{M}$ *is a linear continuous operator, such that*

i) $|\varphi^* v| = |v| \circ \varphi$ *holds* $\mathfrak{m}_\mathrm{Y}$*-a.e. for every* $v \in \mathscr{M}$*,*

ii) *the set* $\{\varphi^* v : v \in \mathscr{M}\}$ *generates* $\varphi^*\mathscr{M}$ *as a module.*

Uniqueness is up to unique isomorphism: given another couple $(\widetilde{\varphi^*\mathscr{M}}, \widetilde{\varphi^*})$ *with the same properties, there is a unique module isomorphism* $\Phi : \varphi^*\mathscr{M} \to \widetilde{\varphi^*\mathscr{M}}$ *such that* $\Phi \circ \varphi^* = \widetilde{\varphi^*}$.

Proof. Uniqueness. We define the space $V \subseteq \varphi^*\mathscr{M}$ of simple elements as

$$V := \left\{ \sum_{i=1}^n \chi_{A_i} \varphi^* v_i \;\middle|\; (A_i)_i \text{ Borel partition of } \mathrm{Y},\ (v_i)_i \subseteq \mathscr{M} \right\}.$$

We are obliged to define $\Phi\big(\sum_i \chi_{A_i}\, \varphi^* v_i\big) := \sum_i \chi_{A_i}\, \widetilde{\varphi^* v_i}$ for any $\sum_i \chi_{A_i}\, \varphi^* v_i \in V$. Since

$$\left| \sum_i \chi_{A_i}\, \widetilde{\varphi^* v_i} \right| = \sum_i \chi_{A_i}\, |\widetilde{\varphi^* v_i}| = \sum_i \chi_{A_i}\, |v_i| \circ \varphi = \sum_i \chi_{A_i}\, |\varphi^* v_i| = \left| \sum_i \chi_{A_i}\, \varphi^* v_i \right| \quad \mathfrak{m}\text{.a.e.},$$

we see that such Φ is well-defined. Moreover, it is also linear and continuous, whence it can be uniquely extended to a map $\Phi : \varphi^*\mathscr{M} \to \widetilde{\varphi^*\mathscr{M}}$. It can be readily proven that Φ is a module isomorphism satisfying $\Phi \circ \varphi^* = \widetilde{\varphi^*}$, thus showing uniqueness.

EXISTENCE. We define the 'pre-pullback module' Ppb as

$$\mathsf{Ppb} := \big\{(A_i, v_i)_{i=1}^n \;\big|\; (A_i)_i \text{ Borel partition of Y}, \, (v_i)_i \subseteq \mathscr{M}\big\}.$$

We consider the following equivalence relation on Ppb: we declare $(A_i, v_i)_i \sim (B_j, w_j)_j$ provided $|v_i - w_j| \circ \varphi = 0$ holds $\mathfrak{m}_{\mathrm{Y}}$-a.e. on $A_i \cap B_j$ for every i, j. We shall denote by $[A_i, v_i]_i$ the equivalence class of $(A_i, v_i)_i$. Hence we introduce some operations on $\mathsf{Ppb}/\sim$:

$$\begin{aligned}
[A_i, v_i]_i + [B_j, w_j]_j &:= [A_i \cap B_j, v_i + w_j]_{i,j},\\
\lambda\,[A_i, v_i]_i &:= [A_i, \lambda\, v_i]_i,\\
\Big(\sum_j \alpha_j\,\chi_{B_j}\Big) \cdot [A_i, v_i]_i &:= [A_i \cap B_j, \alpha_j\, v_i]_{i,j},\\
\big|[A_i, v_i]_i\big| &:= \sum_i \chi_{A_i}\,|v_i| \circ \varphi \in L^2(\mathfrak{m}_{\mathrm{Y}}),\\
\big\|[A_i, v_i]_i\big\| &:= \bigg(\int \big|[A_i, v_i]_i\big|^2 \,\mathrm{d}\mathfrak{m}_{\mathrm{Y}}\bigg)^{1/2}.
\end{aligned}$$

One can prove that $\big(\mathsf{Ppb}/\sim, \|\cdot\|\big)$ is a normed space, then we define $\varphi^*\mathscr{M}$ as its completion and we call $\varphi^*\,\mathscr{M} \to \varphi^*\mathscr{M}$ the map sending any $v \in \mathscr{M}$ to $[\mathrm{Y}, v]$. It can be seen that the above operations can be uniquely extended to $\varphi^*\mathscr{M}$, thus endowing it with the structure of an $L^2(\mathfrak{m}_{\mathrm{Y}})$-normed module, and that $(\varphi^*\mathscr{M}, \varphi^*)$ satisfies the required properties. $\square$

Example 20.2 Consider $\mathscr{M} := L^2(\mathfrak{m}_{\mathrm{X}})$. Then $\varphi^*\mathscr{M} = L^2(\mathfrak{m}_{\mathrm{Y}})$ and $\varphi^* f = f \circ \varphi$ holds for every $f \in L^2(\mathfrak{m}_{\mathrm{X}})$.

Example 20.3 Suppose that we have $\mathrm{Y} = \mathrm{X} \times \mathrm{Z}$, for some metric measure space $(\mathrm{Z}, \mathsf{d}_{\mathrm{Z}}, \mathfrak{m}_{\mathrm{Z}})$ such that $\mathfrak{m}_{\mathrm{Z}}(\mathrm{Z}) < +\infty$. Let us define $\mathsf{d}_{\mathrm{Y}}\big((x_1, z_1), (x_2, z_2)\big)^2 := \mathsf{d}_{\mathrm{X}}(x_1, x_2)^2 + \mathsf{d}_{\mathrm{Z}}(z_1, z_2)^2$ for every pair $(x_1, z_1), (x_2, z_2) \in \mathrm{X} \times \mathrm{Z}$ and $\mathfrak{m}_{\mathrm{Y}} := \mathfrak{m}_{\mathrm{X}} \otimes \mathfrak{m}_{\mathrm{Z}}$. Denote by $\varphi :\, \mathrm{Y} \to \mathrm{X}$ the canonical projection, which has bounded compression as $\varphi_*\mathfrak{m}_{\mathrm{Y}} = \mathfrak{m}_{\mathrm{Z}}(\mathrm{Z})\,\mathfrak{m}_{\mathrm{X}}$.

Now fix an $L^2(\mathfrak{m}_{\mathrm{X}})$-normed module $\mathscr{M}$ and consider the space $L^2(\mathrm{Z}, \mathscr{M})$, which can be naturally endowed with the structure of an $L^2(\mathfrak{m}_Y)$-normed module. For any $f \in L^\infty(\mathfrak{m}_{\mathrm{Y}})$ and $V_{\cdot} \in L^2(\mathrm{Z}, \mathscr{M})$, we have that $f \cdot V_{\cdot} \in L^2(\mathrm{Z}, \mathscr{M})$ is defined as $z \mapsto f(\cdot, z) V_z \in \mathscr{M}$. Given any element $V_{\cdot}$ of $L^2(\mathrm{Z}, \mathscr{M})$, say $z \mapsto V_z$, we have that the pointwise norm $|V_{\cdot}|$ is ($\mathfrak{m}_{\mathrm{Y}}$-a.e.) given by the function $(x, z) \mapsto |V_z|(x)$. Moreover, consider the operator $\hat{\cdot} :\, \mathscr{M} \to L^2(\mathrm{Z}, \mathscr{M})$ sending any $v \in \mathscr{M}$ to the function $\hat{v} :\, \mathrm{Z} \to \mathscr{M}$ that is identically equal to v. We claim that

$$\big(\varphi^*\mathscr{M}, \varphi^*\big) \sim \big(L^2(\mathrm{Z}, \mathscr{M}), \hat{\cdot}\,\big). \tag{20.1}$$

To prove property i) of Theorem 20.1 observe that

$$|\hat{v}_{\cdot}|(x, z) = |V_z|(x) = |v|(x) = \big(|v| \circ \varphi\big)(x, z) \qquad \text{for } \mathfrak{m}_{\mathrm{Y}}\text{-a.e. } (x, z),$$

while ii) follows from density of the simple functions in $L^2(\mathrm{Z}, \mathscr{M})$.

Remark 20.4 Suppose that $\mathfrak{m}_{\mathrm{X}}$ is a Dirac delta. Hence any Banach space $\mathbb{B}$ can be viewed as an $L^2(\mathfrak{m}_{\mathrm{X}})$-normed module (since $L^\infty(\mathfrak{m}_{\mathrm{X}}) \sim \mathbb{R}$). Then it holds that

$$(\varphi^*\mathbb{B}, \varphi^*) \sim \big(L^2(\mathrm{Z}, \mathbb{B}), \hat{\cdot}\,\big) \tag{20.2}$$

as a consequence of the previous example. ■

Example 20.5 Fix an $L^2(\mathfrak{m}_{\mathrm{X}})$-normed module $\mathscr{M}$. Suppose that the space Y is a subset of X with $\mathfrak{m}_{\mathrm{X}}(\mathrm{Y}) > 0$. Call $\varphi : \mathrm{Y} \to \mathrm{X}$ the inclusion map, which has bounded compression provided Y is equipped with the measure $\mathfrak{m}_{\mathrm{Y}} := \mathfrak{m}_{\mathrm{X}}|_{\mathrm{Y}}$. The $L^2(\mathfrak{m}_{\mathrm{Y}})$-normed module $\mathscr{M}|_{\mathrm{Y}}$ is defined as $\mathscr{M}|_{\mathrm{Y}} := \mathscr{M}/\sim$, where $v \sim w$ if and only if $|v - w| = 0$ holds $\mathfrak{m}_{\mathrm{X}}$-a.e. on Y. Then

$$(\varphi^*\mathscr{M}, \varphi^*) \sim \big(\mathscr{M}|_{\mathrm{Y}}, \pi\big), \tag{20.3}$$

where $\pi : \mathscr{M} \to \mathscr{M}|_{\mathrm{Y}}$ is the canonical projection.

Proposition 20.6 *Let* $(\mathrm{X}, \mathsf{d}_{\mathrm{X}}, \mathfrak{m}_{\mathrm{X}})$, $(\mathrm{Y}, \mathsf{d}_{\mathrm{Y}}, \mathfrak{m}_{\mathrm{Y}})$ *be metric measure spaces. Let* $\varphi : \mathrm{Y} \to \mathrm{X}$ *be a map of bounded compression and* $\mathscr{M}$ *an* $L^2(\mathfrak{m}_{\mathrm{X}})$*-normed module. Consider a generating linear subspace* V *of* $\mathscr{M}$*. Let* $\mathscr{N}$ *be an* $L^2(\mathfrak{m}_{\mathrm{Y}})$*-normed module and* $T : V \to \mathscr{N}$ *a linear map satisfying the inequality*

$$\big|T(v)\big| \leq C\,|v| \circ \varphi \quad \mathfrak{m}_{\mathrm{Y}}\text{-}a.e. \qquad \textit{for every } v \in V, \tag{20.4}$$

for some constant $C > 0$*. Then there is a unique linear continuous extension* $\hat{T} : \mathscr{M} \to \mathscr{N}$ *of* T *such that* $\big|\hat{T}(v)\big| \leq C\,|v| \circ \varphi$ *holds* $\mathfrak{m}_{\mathrm{Y}}$*-a.e. for every* $v \in \mathscr{M}$*.*

Proof. First of all, we claim that any extension $\hat{T}$ as in the thesis must satisfy

$$\hat{T}(\chi_A\, v) = \chi_A \circ \varphi\, T(v) \qquad \text{for every } v \in V \text{ and } A \subseteq \mathrm{X} \text{ Borel.} \tag{20.5}$$

To prove the claim, observe that

$$\hat{T}(\chi_A\, v) + \hat{T}(\chi_{A^c}\, v) = T(v) = \chi_A \circ \varphi\, T(v) + \chi_{A^c} \circ \varphi\, T(v). \tag{20.6}$$

Moreover, we have that $\chi_A \circ \varphi\,\big|\hat{T}(\chi_{A^c}\, v)\big| \leq C\,\chi_A \circ \varphi\,|\chi_{A^c}\, v| \circ \varphi = 0$, i.e. $\chi_A \circ \varphi\,\hat{T}(\chi_{A^c}\, v) = 0$. Similarly, one has that $\chi_{A^c} \circ \varphi\,\hat{T}(\chi_A\, v) = 0$. Hence by multiplying both sides of (20.6) by the function $\chi_A \circ \varphi$ we get $\chi_A \circ \varphi\,\hat{T}(\chi_A\, v) = \chi_A \circ \varphi\, T(v)$ and accordingly

$$\hat{T}(\chi_A\, v) = \chi_A \circ \varphi\,\hat{T}(\chi_A\, v) + \chi_{A^c} \circ \varphi\,\hat{T}(\chi_A\, v) = \chi_A \circ \varphi\,\hat{T}(\chi_A\, v) = \chi_A \circ \varphi\, T(v),$$

thus proving the validity of (20.5).

In light of (20.5), we necessarily have to define $\hat{T}\big(\sum_i \chi_{A_i}\, v_i\big) := \sum_i \chi_{A_i} \circ \varphi\, T(v_i)$ for any finite Borel partition $(A_i)_i$ of X and for any $(v_i)_i \subseteq V$. Well-posedness of such definition stems from the $\mathfrak{m}_{\mathrm{Y}}$-a.e. inequality

$$\Big|\sum_i \chi_{A_i} \circ \varphi\, T(v_i)\Big| = \sum_i \chi_{\varphi^{-1}(A_i)}\,\big|T(v_i)\big| \leq C \sum_i \big(\chi_{A_i}\,|v_i|\big) \circ \varphi = C\,\Big|\sum_i \chi_{A_i}\, v_i\Big| \circ \varphi,$$

which also grants (linearity and) continuity of $\hat{T}$. Therefore the operator $\hat{T}$ admits a unique extension $\hat{T} : \mathscr{M} \to \mathscr{N}$ with the required properties. □

Remark 20.7 The operator $\hat{T}$ in Proposition 20.6 also satisfies

$$\hat{T}(f\,v) = f \circ \varphi\,\hat{T}(v) \quad \text{for every } f \in L^\infty(\mathfrak{m}_{\mathrm{X}}) \text{ and } v \in \mathscr{M}. \tag{20.7}$$

Such property can be easily obtained by means of an approximation argument. ■

The ideas contained in the proof of Proposition 20.6 can be adapted to show the following result, whose proof will be omitted.

Proposition 20.8 *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space. Let* $\mathscr{M}_1$, $\mathscr{M}_2$ *be* $L^2(\mathfrak{m})$*-normed modules and* $T : \mathscr{M}_1 \to \mathscr{M}_2$ *a linear map such that*

$$\big|T(v)\big| \leq C\,|v| \quad \mathfrak{m}\text{-a.e.} \qquad \text{for every } v \in \mathscr{M}_1, \tag{20.8}$$

for some constant $C > 0$*. Then* T *is* $L^\infty(\mathfrak{m})$*-linear and continuous.*

Exercise 20.9 Let $T : L^2(\mathfrak{m}) \to L^2(\mathfrak{m})$ be an $L^\infty(\mathfrak{m})$-linear and continuous operator. Prove that there exists a unique $g \in L^\infty(\mathfrak{m})$ such that $T(f) = gf$ for every $f \in L^2(\mathfrak{m})$.

Theorem 20.10 (Universal property) *Let* $(\mathrm{X}, \mathsf{d}_{\mathrm{X}}, \mathfrak{m}_{\mathrm{X}})$, $(\mathrm{Y}, \mathsf{d}_{\mathrm{Y}}, \mathfrak{m}_{\mathrm{Y}})$ *be two metric measure spaces. Let* $\varphi : \mathrm{Y} \to \mathrm{X}$ *be a map of bounded compression. Consider an* $L^2(\mathfrak{m}_{\mathrm{X}})$*-normed module* $\mathscr{M}$*, an* $L^2(\mathfrak{m}_{\mathrm{Y}})$*-normed module* $\mathscr{N}$ *and a linear map* $T : \mathscr{M} \to \mathscr{N}$*. Suppose that there exists a constant* $C > 0$ *such that*

$$\big|T(v)\big| \leq C\,|v| \circ \varphi \quad \mathfrak{m}_{\mathrm{Y}}\text{-a.e.} \qquad \text{for every } v \in \mathscr{M}. \tag{20.9}$$

Then there exists a unique $L^\infty(\mathfrak{m}_Y)$*-linear continuous operator* $\hat{T} : \varphi^*\mathscr{M} \to \mathscr{N}$*, called* lifting *of* T*, such that* $\big|\hat{T}(w)\big| \leq C\,|w|$ *holds* $\mathfrak{m}_{\mathrm{Y}}$*-a.e. for any* $w \in \varphi^*\mathscr{M}$ *and such that*

$$\begin{tikzcd} \mathscr{M} \arrow[r, "\varphi^*"] \arrow[dr, "T"'] & \varphi^*\mathscr{M} \arrow[d, "\hat{T}"] \\ & \mathscr{N} \end{tikzcd} \tag{20.10}$$

is a commutative diagram.

Proof. Call $V := \{\varphi^* v \,:\, v \in \mathscr{M}\}$, then V is a generating linear subspace of $\varphi^*\mathscr{M}$. We define the map $S : V \to \mathscr{N}$ as $S(\varphi^* v) := T(v)$ for every $v \in \mathscr{M}$. The $\mathfrak{m}_{\mathrm{Y}}$-a.e. inequality

$$\big|T(v)\big| \leq C\,|v| \circ \varphi = C\,|\varphi^* v|$$

grants that S is well-defined. Hence Proposition 20.6 guarantees that S admits a unique extension $\hat{T} : \varphi^*\mathscr{M} \to \mathscr{N}$ with the required properties. □

21 Lesson [24/01/2018]

Theorem 21.1 (Functoriality) *Let* $(\mathrm{X},\mathsf{d}_\mathrm{X},\mathfrak{m}_\mathrm{X})$, $(\mathrm{Y},\mathsf{d}_\mathrm{Y},\mathfrak{m}_\mathrm{Y})$ *and* $(\mathrm{Z},\mathsf{d}_\mathrm{Z},\mathfrak{m}_\mathrm{Z})$ *be metric measure spaces. Let* $\varphi:\mathrm{Y}\to\mathrm{X}$ *and* $\psi:\mathrm{Z}\to\mathrm{Y}$ *be maps of bounded compression. Fix an* $L^2(\mathfrak{m}_\mathrm{X})$*-normed module* $\mathscr{M}$*. Then the map* $\varphi\circ\psi$ *has bounded compression and*

$$\big(\psi^*(\varphi^*\mathscr{M}),\psi^*\circ\varphi^*\big)\sim\big((\varphi\circ\psi)^*\mathscr{M},(\varphi\circ\psi)^*\big). \tag{21.1}$$

Proof. It is trivial to check that $\varphi\circ\psi$ has bounded compression. It only remains to show that

$$\begin{aligned}&\big|\psi^*(\varphi^*v)\big|=|v|\circ\varphi\circ\psi \quad \mathfrak{m}_\mathrm{Z}\text{-a.e.}\quad \text{for every } v\in\mathscr{M},\\ &\big\{\psi^*(\varphi^*v)\,:\,v\in\mathscr{M}\big\}\quad \text{generates } \psi^*(\varphi^*\mathscr{M}) \text{ as a module.}\end{aligned}$$

To prove the former, just notice that $\big|\psi^*(\varphi^*v)\big|=|\varphi^*v|\circ\psi=|v|\circ\varphi\circ\psi$. For the latter, notice that the set V of all finite sums of the form $\sum_i\chi_{A_i}\,\varphi^*v_i$, with $(A_i)_i$ Borel partition of Y and $(v_i)_i\subseteq\mathscr{M}$, is a dense vector subspace of $\varphi^*\mathscr{M}$. Hence the set of all finite sums of the form $\sum_j\chi_{B_j}\,\psi^*w_j$, with $(B_j)_j$ Borel partition of Z and $(w_j)_j\subseteq V$, is dense in $\psi^*(\varphi^*\mathscr{M})$, thus proving that $\big\{\psi^*(\varphi^*v)\,:\,v\in\mathscr{M}\big\}$ generates $\psi^*(\varphi^*\mathscr{M})$. □

We now investigate the relation between $(\varphi^*\mathscr{M})^*$ and $\varphi^*\mathscr{M}^*$. Under suitable assumptions, it will turn out that the operations of taking the dual and passing to the pullback commute.

Proposition 21.2 *Let* $(\mathrm{X},\mathsf{d}_\mathrm{X},\mathfrak{m}_\mathrm{X})$, $(\mathrm{Y},\mathsf{d}_\mathrm{Y},\mathfrak{m}_\mathrm{Y})$ *be metric measure spaces and* $\varphi:\mathrm{Y}\to\mathrm{X}$ *a map of bounded compression. Then there exists a unique* $L^\infty(\mathfrak{m}_\mathrm{Y})$*-bilinear and continuous map* $B:\varphi^*\mathscr{M}\times\varphi^*\mathscr{M}^*\to L^1(\mathfrak{m}_\mathrm{Y})$ *such that* $B(\varphi^*v,\varphi^*L)=L(v)\circ\varphi$ *is satisfied* $\mathfrak{m}_\mathrm{Y}$*-a.e. for every* $v\in\mathscr{M}$ *and* $L\in\mathscr{M}^*$.

Proof. We are forced to declare $B\big(\sum_i\chi_{E_i}\,\varphi^*v_i,\sum_j\chi_{F_j}\,\varphi^*L_j\big):=\sum_{i,j}\chi_{E_i\cap F_j}\,L_j(v_i)\circ\varphi$. Since

$$\begin{aligned}\bigg|\sum_{i,j}\chi_{E_i\cap F_j}\,L_j(v_i)\circ\varphi\bigg|&=\sum_{i,j}\chi_{E_i\cap F_j}\,\big|L_j(v_i)\big|\circ\varphi\le\sum_{i,j}\chi_{E_i\cap F_j}\,|L_j|\circ\varphi\,|v_i|\circ\varphi\\ &=\bigg(\sum_i\chi_{E_i}\,|v_i|\circ\varphi\bigg)\bigg(\sum_j\chi_{F_j}\,|L_j|\circ\varphi\bigg)\\ &=\bigg|\sum_i\chi_{E_i}\,\varphi^*v_i\bigg|\bigg|\sum_j\chi_{F_j}\,\varphi^*L_j\bigg|,\end{aligned}$$

we see that B is (well-defined and) continuous, whence it can be uniquely extended to an operator $B:\varphi^*\mathscr{M}\times\varphi^*\mathscr{M}^*\to L^1(\mathfrak{m}_\mathrm{Y})$ satisfying all of the required properties. □

Proposition 21.3 *Under the assumptions of Proposition 21.2, the map*

$$I:\varphi^*\mathscr{M}^*\longrightarrow(\varphi^*\mathscr{M})^*,\quad W\longmapsto B(\cdot,W) \tag{21.2}$$

is well-defined, $L^\infty(\mathfrak{m}_\mathrm{Y})$*-linear continuous and preserving the pointwise norm, i.e. the* $\mathfrak{m}_\mathrm{Y}$*-a.e. equality* $\big|I(W)\big|=|W|$ *holds for every* $W\in\varphi^*\mathscr{M}^*$.

Proof. The map $I(W) : \varphi^*\mathscr{M} \to L^1(\mathfrak{m}_Y)$ is $L^\infty(\mathfrak{m}_Y)$-linear continuous by Proposition 21.2, in other words $I(W) \in (\varphi^*\mathscr{M})^*$, which shows that I is well-posed. Moreover, notice that

$$\big|I(W)\big| = \operatorname*{ess\,sup}_{\substack{V\in\varphi^*\mathscr{M},\\ |V|\leq 1\ \mathfrak{m}_Y\text{-a.e.}}} \big|B(V,W)\big| \leq \operatorname*{ess\,sup}_{\substack{V\in\varphi^*\mathscr{M},\\ |V|\leq 1\ \mathfrak{m}_Y\text{-a.e.}}} |V||W| \leq |W| \quad \mathfrak{m}_Y\text{-a.e.},$$

whence I can be easily proven to be $L^\infty(\mathfrak{m}_Y)$-linear and continuous. Finally, to conclude it suffices to prove that also $\big|I(W)\big| \geq |W|$ holds $\mathfrak{m}_Y$-a.e. in Y. By density, it is actually enough to obtain it for W of the form $\sum_{j=1}^n \chi_{F_j}\,\varphi^*L_j$. Then observe that

$$\begin{aligned}\big|I(W)\big| &\geq \operatorname*{ess\,sup}_{\substack{v_1,\dots,v_n\in\mathscr{M},\\ |v_1|,\dots,|v_n|\leq 1\ \mathfrak{m}_X\text{-a.e.}}} I(W)\bigg(\sum_{j=1}^n \chi_{F_j}\,\varphi^*v_j\bigg) = \sum_{j=1}^n \chi_{F_j} \operatorname*{ess\,sup}_{\substack{v_j\in\mathscr{M},\\ |v_j|\leq 1\ \mathfrak{m}_X\text{-a.e.}}} L_j(v_j)\circ\varphi \\ &= \sum_{j=1}^n \chi_{F_j}\,|L_j|\circ\varphi = \sum_{j=1}^n \chi_{F_j}\,|\varphi^*L_j| = |W|\end{aligned}$$

holds $\mathfrak{m}_Y$-a.e. in Y. Therefore the statement is achieved. □

Remark 21.4 In particular, Proposition 21.3 shows that the map I is an isometric embedding of $\varphi^*\mathscr{M}^*$ into $(\varphi^*\mathscr{M})^*$. However, as we are going to show in the next example, the operator I needs not be surjective. ■

Example 21.5 Suppose that $\mathrm{X} := \{\bar{x}\}$ and $\mathfrak{m}_X := \delta_{\bar{x}}$. Moreover, let $\mathrm{Y} := [0,1]$ be endowed with the Lebesgue measure and denote by φ the unique map from Y to X, which is clearly of bounded compression. Since $L^\infty(\mathfrak{m}_X) \sim \mathbb{R}$, we can view any Banach space $\mathbb{B}$ as an $L^2(\mathfrak{m}_X)$-normed module, so that Remark 20.4 yields

$$\begin{aligned}(\varphi^*\mathbb{B})^* &\sim \big(L^2([0,1],\mathbb{B})\big)',\\ \varphi^*\mathbb{B}^* &\sim L^2([0,1],\mathbb{B}').\end{aligned}$$

In general, $L^2([0,1],\mathbb{B}')$ is only embedded into $\big(L^2([0,1],\mathbb{B})\big)'$, via the map that sends any element $\ell_. \in L^2([0,1],\mathbb{B}')$ to $L^2([0,1],\mathbb{B}) \ni v_. \mapsto \int_0^1 \ell_t(v_t)\,\mathrm{d}t$, which clearly belongs to the space $\big(L^2([0,1],\mathbb{B})\big)'$. Now consider e.g. the case in which $\mathbb{B} := L^1(0,1)$. Let us define the map $T :\ L^2\big([0,1],L^1(0,1)\big) \to \mathbb{R}$ as

$$T(f) := \int_0^1\int_0^1 f_t(x)\,g_t(x)\,\mathrm{d}x\,\mathrm{d}t \quad \text{ for every } f \in L^2\big([0,1],L^1(0,1)\big),$$

where $g_t := \chi_{[0,t]}$. Hence T does not come from any element of $L^2\big([0,1],L^\infty(0,1)\big)$: it should come from the map $t \mapsto g_t \in L^\infty(0,1)$, which is not essentially separably valued. This shows that $L^2\big([0,1],L^\infty(0,1)\big)$ and the dual of $L^2\big([0,1],L^1(0,1)\big)$ are different.

Lemma 21.6 *Let* $(\mathrm{X},\mathsf{d}_X,\mathfrak{m}_X)$, $(\mathrm{Y},\mathsf{d}_Y,\mathfrak{m}_Y)$ *be metric measure spaces and* $\varphi : \mathrm{Y} \to \mathrm{X}$ *a map of bounded compression. Let* $\mathscr{H}$ *be a Hilbert module on* X. *Then* $\varphi^*\mathscr{H}$ *is a Hilbert module as well.*

Proof. Notice that

$$2\big(|\varphi^*v|^2+|\varphi^*w|^2\big)=2\big(|v|^2+|w|^2\big)\circ\varphi=|v+w|^2\circ\varphi+|v-w|^2\circ\varphi=|\varphi^*v+\varphi^*w|^2+|\varphi^*v-\varphi^*w|^2$$

is satisfied $\mathfrak{m}_{\mathrm{Y}}$-a.e. for any $v,w\in\mathscr{H}$. Then the pointwise parallelogram identity can be shown to hold for elements of the form $\sum_i\chi_{E_i}\,\varphi^*v_i$, thus accordingly for all elements of $\varphi^*\mathscr{H}$ by an approximation argument. This proves that $\varphi^*\mathscr{H}$ is a Hilbert module, as required. □

Proposition 21.7 *Let* $(\mathrm{X},\mathsf{d}_{\mathrm{X}},\mathfrak{m}_{\mathrm{X}})$, $(\mathrm{Y},\mathsf{d}_{\mathrm{Y}},\mathfrak{m}_{\mathrm{Y}})$ *be metric measure spaces and* $\varphi:\mathrm{Y}\to\mathrm{X}$ *a map of bounded compression. Let* $\mathscr{H}$ *be a Hilbert module on* X. *Then*

$$\varphi^*\mathscr{H}^*\sim(\varphi^*\mathscr{H})^*. \tag{21.3}$$

Proof. Consider the map $I:\varphi^*\mathscr{H}^*\to(\varphi^*\mathscr{H})^*$ of Proposition 21.3. We aim to prove that I is surjective. Denote by $\mathscr{R}:\mathscr{H}\to\mathscr{H}^*$ and $\widehat{\mathscr{R}}:\varphi^*\mathscr{H}\to(\varphi^*\mathscr{H})^*$ the Riesz isomorphisms, as in Theorem 18.11. Note that $\varphi^*\circ\mathscr{R}:\mathscr{H}\to\varphi^*\mathscr{H}^*$ satisfies $\big|(\varphi^*\circ\mathscr{R})(v)\big|=|v|\circ\varphi$ $\mathfrak{m}_{\mathrm{Y}}$-a.e. for any $v\in\mathscr{H}$, whence Theorem 20.10 grants that there exists a unique $L^\infty(\mathfrak{m}_{\mathrm{Y}})$-linear continuous operator $\widehat{\varphi^*\circ\mathscr{R}}:\varphi^*\mathscr{H}\to\varphi^*\mathscr{H}^*$ such that $\widehat{\varphi^*\circ\mathscr{R}}(\varphi^*v)=(\varphi^*\circ\mathscr{R})(v)$ holds for every $v\in\mathscr{H}$. Now let us define $J:=\widehat{\varphi^*\circ\mathscr{R}}\circ\widehat{\mathscr{R}}^{-1}:(\varphi^*\mathscr{H})^*\to\varphi^*\mathscr{H}^*$. We claim that

$$I\circ J=\mathrm{id}_{(\varphi^*\mathscr{H})^*}. \tag{21.4}$$

Given that $I\circ J$ is $L^\infty(\mathfrak{m}_{\mathrm{Y}})$-linear continuous by construction, it suffices to check that $I\circ J$ is the identity on the subspace $\big\{\widehat{\mathscr{R}}(\varphi^*v)\,:\,v\in\mathscr{H}\big\}$, which generates $(\varphi^*\mathscr{H})^*$ as a module. Observe that for any $v,w\in\mathscr{H}$ it holds that

$$\begin{aligned}\widehat{\mathscr{R}}(\varphi^*v)(\varphi^*w)&=\langle\varphi^*v,\varphi^*w\rangle=\langle v,w\rangle\circ\varphi,\\(I\circ J)\big(\widehat{\mathscr{R}}(\varphi^*v)\big)(\varphi^*w)&=I\big(\widehat{\varphi^*\circ\mathscr{R}}(\varphi^*v)\big)(\varphi^*w)=I\big((\varphi^*\circ\mathscr{R})(v)\big)(\varphi^*w)=\big(\mathscr{R}(v)(w)\big)\circ\varphi\\&=\langle v,w\rangle\circ\varphi,\end{aligned}$$

whence (21.4) follows. This grants that I is surjective, thus concluding the proof. □

Remark 21.8 Suppose that the map $\varphi:\mathrm{Y}\to\mathrm{X}$ is invertible and both φ, φ^{-1} have bounded compression. Then Theorem 21.1 grants that $(\varphi^{-1})^*(\varphi^*\mathscr{M})\sim\mathscr{M}$, thus in particular one has that $\varphi^*:\mathscr{M}\to\varphi^*\mathscr{M}$ is bijective. Hence, morally speaking, $\mathscr{M}$ and $\varphi^*\mathscr{M}$ are the same module, up to identifying $L^\infty(\mathfrak{m}_{\mathrm{X}})$ and $L^\infty(\mathfrak{m}_{\mathrm{Y}})$ via invertible map $f\mapsto f\circ\varphi$. ■

Definition 21.9 (Maps of bounded deformation) *Let* $(\mathrm{X},\mathsf{d}_{\mathrm{X}},\mathfrak{m}_{\mathrm{X}})$, $(\mathrm{Y},\mathsf{d}_{\mathrm{Y}},\mathfrak{m}_{\mathrm{Y}})$ *be metric measure spaces. Then a map* $\varphi:\mathrm{Y}\to\mathrm{X}$ *is said to be* of bounded deformation *provided it is Lipschitz and of bounded compression.*

A map of bounded deformation $\varphi:\mathrm{Y}\to\mathrm{X}$ naturally induces a map

$$\begin{aligned}\boldsymbol{\varphi}:\,C\big([0,1],\mathrm{Y}\big)&\longrightarrow C\big([0,1],\mathrm{X}\big),\\\gamma&\longmapsto\varphi\circ\gamma.\end{aligned} \tag{21.5}$$

It is then easy to prove that

$$\gamma \text{ is an AC curve in Y} \quad \Longrightarrow \quad \begin{array}{l} \boldsymbol{\varphi}(\gamma) \text{ is an AC curve in X and} \\ \big|\dot{\boldsymbol{\varphi}(\gamma)}_t\big| \leq \mathrm{Lip}(\varphi)\,|\dot\gamma_t| \text{ for a.e. } t. \end{array} \tag{21.6}$$

Indeed, we have $\mathsf{d}_{\mathrm{X}}\big(\varphi(\gamma_t),\varphi(\gamma_s)\big) \leq \mathrm{Lip}(\varphi)\,\mathsf{d}_{\mathrm{Y}}(\gamma_t,\gamma_s) \leq \mathrm{Lip}(\varphi)\int_s^t |\dot\gamma_r|\,\mathrm{d}f$ for all $s<t$.

Lemma 21.10 *Let $\boldsymbol{\pi}$ be a test plan on* Y *and $\varphi:\,\mathrm{Y}\to\mathrm{X}$ a map of bounded deformation. Then $\boldsymbol{\varphi}_*\boldsymbol{\pi}$ is a test plan on* X.

Proof. Observe that

$$\begin{gathered}
(\mathrm{e}_t)_*\boldsymbol{\varphi}_*\boldsymbol{\pi} = \varphi_*(\mathrm{e}_t)_*\boldsymbol{\pi} \leq \varphi_*(C\,\mathfrak{m}_{\mathrm{Y}}) \leq \mathrm{Comp}(\varphi)\,C\,\mathfrak{m}_{\mathrm{X}} \quad \text{ for every } t\in[0,1],\\
\int_0^1\!\!\int |\dot\gamma_t|^2\,\mathrm{d}\boldsymbol{\varphi}_*\boldsymbol{\pi}(\gamma)\,\mathrm{d}t = \int_0^1\!\!\int \big|\dot{\boldsymbol{\varphi}(\gamma)}_t\big|^2\,\mathrm{d}\boldsymbol{\pi}(\gamma)\,\mathrm{d}t \leq \mathrm{Lip}(\varphi)^2\int_0^1\!\!\int |\dot\gamma_t|^2\,\mathrm{d}\boldsymbol{\pi}(\gamma)\,\mathrm{d}t < +\infty,
\end{gathered}$$

whence the statement follows. □

By duality with Lemma 21.10, we can thus obtain the following result:

Proposition 21.11 *Let $\varphi:\,\mathrm{Y}\to\mathrm{X}$ be a map of bounded deformation and $f\in\mathrm{S}^2(\mathrm{X})$. Then it holds that $f\circ\varphi\in\mathrm{S}^2(\mathrm{Y})$ and*

$$\big|D(f\circ\varphi)\big| \leq \mathrm{Lip}(\varphi)\,|Df|\circ\varphi \quad \textit{holds } \mathfrak{m}_{\mathrm{Y}}\textit{-a.e. in } \mathrm{Y}. \tag{21.7}$$

Proof. Since $|Df|\circ\varphi\in L^2(\mathfrak{m}_{\mathrm{Y}})$, it only suffices to prove that $\mathrm{Lip}(\varphi)\,|Df|\circ\varphi$ is a weak upper gradient for f. Then fix any test plan $\boldsymbol{\pi}$ on Y. We have that

$$\begin{aligned}
\int \big|f\circ\varphi\circ\mathrm{e}_1 - f\circ\varphi\circ\mathrm{e}_0\big|\,\mathrm{d}\boldsymbol{\pi} = \int |f\circ\mathrm{e}_1 - f\circ\mathrm{e}_0|\,\mathrm{d}\boldsymbol{\varphi}_*\boldsymbol{\pi} &\leq \int_0^1\!\!\int |Df|(\gamma_t)\,|\dot\gamma_t|\,\mathrm{d}\boldsymbol{\varphi}_*\boldsymbol{\pi}(\gamma)\,\mathrm{d}t\\
&= \int_0^1\!\!\int |Df|\big(\boldsymbol{\varphi}(\gamma)_t\big)\,\big|\dot{\boldsymbol{\varphi}(\gamma)}_t\big|\,\mathrm{d}\boldsymbol{\pi}(\gamma)\,\mathrm{d}t\\
&\leq \mathrm{Lip}(\varphi)\int_0^1\!\!\int \big(|Df|\circ\varphi\big)(\gamma_t)\,|\dot\gamma_t|\,\mathrm{d}\boldsymbol{\pi}(\gamma)\,\mathrm{d}t,
\end{aligned}$$

proving that $\mathrm{Lip}(\varphi)\,|Df|\circ\varphi$ is a weak upper gradient, as required. □

Theorem 21.12 (Pullback of 1-forms) *Let $(\mathrm{X},\mathsf{d}_{\mathrm{X}},\mathfrak{m}_{\mathrm{X}})$, $(\mathrm{Y},\mathsf{d}_{\mathrm{Y}},\mathfrak{m}_{\mathrm{Y}})$ be metric measure spaces and $\varphi:\,\mathrm{Y}\to\mathrm{X}$ a map of bounded deformation. Then there exists a unique linear and continuous operator $\varphi^*:\,L^2(T^*\mathrm{X})\to L^2(T^*\mathrm{Y})$ such that*

$$\begin{aligned}
\varphi^*\mathrm{d}f &= \mathrm{d}(f\circ\varphi) \quad &&\textit{for every } f\in\mathrm{S}^2(\mathrm{X}),\\
\varphi^*(g\,\omega) &= g\circ\varphi\,\varphi^*\omega \quad &&\textit{for every } g\in L^\infty(\mathfrak{m}_{\mathrm{X}}) \textit{ and } \omega\in L^2(T^*\mathrm{X}).
\end{aligned} \tag{21.8}$$

Moreover, it holds that

$$|\varphi^*\omega| \leq \mathrm{Lip}(\varphi)\,|\omega|\circ\varphi \quad \mathfrak{m}_{\mathrm{Y}}\textit{-a.e.} \quad \textit{for every } \omega\in L^2(T^*\mathrm{X}). \tag{21.9}$$

Proof. We are obliged to define $\varphi^*\big(\sum_i \chi_{E_i}\,\mathrm{d}f_i\big) := \sum_i \chi_{E_i}\circ\varphi\,\mathrm{d}(f_i\circ\varphi)$. Given that

$$\Big|\sum_i \chi_{E_i}\circ\varphi\,\mathrm{d}(f_i\circ\varphi)\Big| = \sum_i \chi_{\varphi^{-1}(E_i)}\,\big|\mathrm{d}(f_i\circ\varphi)\big| \overset{(21.7)}{\leq} \mathrm{Lip}(\varphi)\sum_i \chi_{\varphi^{-1}(E_i)}\,|\mathrm{d}f_i|\circ\varphi$$
$$= \mathrm{Lip}(\varphi)\,\Big|\sum_i \chi_{E_i}\,\mathrm{d}f_i\Big|,$$

we see that φ^* is well-defined, linear and continuous, then it can be uniquely extended to an operator $\varphi^* : L^2(T^*\mathrm{X}) \to L^2(T^*\mathrm{Y})$ having all the required properties. $\square$

We have introduced two different notions of pullback for the cotangent module $L^2(T^*\mathrm{X})$. We shall make use of the notation $\varphi^* : L^2(T^*\mathrm{X}) \to L^2(T^*\mathrm{Y})$ for the pullback described in Theorem 21.12, while we write $[\varphi^*] : L^2(T^*\mathrm{X}) \to \varphi^*L^2(T^*\mathrm{X})$ for the one of Theorem 20.1.

22 Lesson [29/01/2018]

Theorem 22.1 (Differential of a map of bounded deformation) *Let us consider two metric measure spaces* $(\mathrm{X},\mathsf{d}_\mathrm{X},\mathfrak{m}_\mathrm{X})$ *and* $(\mathrm{Y},\mathsf{d}_\mathrm{Y},\mathfrak{m}_\mathrm{Y})$. *Suppose* $(\mathrm{X},\mathsf{d}_\mathrm{X},\mathfrak{m}_\mathrm{X})$ *is infinitesimally Hilbertian. Let* $\varphi : \mathrm{Y} \to \mathrm{X}$ *be a map of bounded deformation. Then there exists a unique* $L^\infty(\mathfrak{m}_\mathrm{Y})$*-linear continuous map* $\mathrm{d}\varphi : L^2(T\mathrm{Y}) \to \varphi^*L^2(T\mathrm{X})$*, called* differential *of* φ*, such that*

$$[\varphi^*\omega]\big(\mathrm{d}\varphi(v)\big) = \varphi^*\omega(v) \quad \text{for every } v \in L^2(T\mathrm{Y}) \text{ and } \omega \in L^2(T^*\mathrm{X}). \tag{22.1}$$

Moreover, it holds that

$$\big|\mathrm{d}\varphi(v)\big| \leq \mathrm{Lip}(\varphi)\,|v| \quad \mathfrak{m}_\mathrm{Y}\text{-a.e.} \qquad \text{for every } v \in L^2(T\mathrm{Y}). \tag{22.2}$$

Proof. Denote by V the generating linear subspace $\big\{[\varphi^*\omega] : \omega \in L^2(T^*\mathrm{X})\big\}$ of $\varphi^*L^2(T^*\mathrm{X})$. Fix $v \in L^2(T\mathrm{Y})$ and define $L_v : V \to L^1(\mathfrak{m}_\mathrm{Y})$ as $L_v[\varphi^*\omega] := \varphi^*\omega(v)$. The $\mathfrak{m}_\mathrm{Y}$-a.e. inequality

$$\big|\varphi^*\omega(v)\big| \leq |\varphi^*\omega|\,|v| \overset{(21.9)}{\leq} \mathrm{Lip}(\varphi)\,|\omega|\circ\varphi\,|v| = \mathrm{Lip}(\varphi)\,|v|\,\big|[\varphi^*\omega]\big| \tag{22.3}$$

grants that L_v is a well-defined, linear and continuous operator. Hence there exists a unique element $\mathrm{d}\varphi(v) \in \big(\varphi^*L^2(T^*\mathrm{X})\big)^* \sim \varphi^*L^2(T\mathrm{X})$ such that $[\varphi^*\omega]\big(\mathrm{d}\varphi(v)\big) = \varphi^*\omega(v)$. Moreover, such element necessarily satisfies $\big|\mathrm{d}\varphi(v)\big| \leq \mathrm{Lip}(\varphi)\,|v|$ $\mathfrak{m}_\mathrm{Y}$-a.e. in Y. Thus to conclude it only remains to show that the assignment $L^2(T\mathrm{Y}) \ni v \mapsto \mathrm{d}\varphi(v) \in \varphi^*L^2(T\mathrm{X})$ is $L^\infty(\mathfrak{m}_\mathrm{Y})$-linear. This follows from the chain of equalities

$$[\varphi^*\omega]\big(\mathrm{d}\varphi(f\,v)\big) = \varphi^*\omega(f\,v) = f\,\varphi^*\omega(v) = f\,[\varphi^*\omega]\big(\mathrm{d}\varphi(v)\big),$$

which holds $\mathfrak{m}_\mathrm{Y}$-a.e. for every choice of $f \in L^\infty(\mathfrak{m}_\mathrm{Y})$ and $v \in L^2(T\mathrm{Y})$. $\square$

In the case in which φ is invertible and its inverse is a map of bounded compression, we have an alternative definition of differential:

Theorem 22.2 *Let* $(\mathrm{X},\mathsf{d}_{\mathrm{X}},\mathfrak{m}_{\mathrm{X}})$, $(\mathrm{Y},\mathsf{d}_{\mathrm{Y}},\mathfrak{m}_{\mathrm{Y}})$ *be metric measure spaces and let* $\varphi:\mathrm{Y}\to\mathrm{X}$ *be a map of bounded deformation. Suppose that* φ *is invertible and that* φ^{-1} *has bounded compression. Then there exists a unique linear continuous operator* $\mathrm{d}\varphi: L^2(T\mathrm{Y})\to L^2(T\mathrm{X})$ *such that*

$$\omega\big(\mathrm{d}\varphi(v)\big)=\big(\varphi^*\omega(v)\big)\circ\varphi^{-1}\quad \mathfrak{m}_{\mathrm{X}}\text{-a.e.}\quad \text{for every } v\in L^2(T\mathrm{Y}) \text{ and } \omega\in L^2(T^*\mathrm{X}). \tag{22.4}$$

Moreover, it holds that

$$\big|\mathrm{d}\varphi(v)\big|\leq \mathrm{Lip}(\varphi)\,|v|\circ\varphi^{-1}\quad \mathfrak{m}_{\mathrm{X}}\text{-a.e.}\quad \text{for every } v\in L^2(T\mathrm{Y}). \tag{22.5}$$

Proof. Fix $v\in L^2(T\mathrm{Y})$. Denote by $\mathrm{d}\varphi(v)$ the map $L^2(T^*\mathrm{X})\ni\omega\mapsto\big(\varphi^*\omega(v)\big)\circ\varphi^{-1}\in L^1(\mathfrak{m}_{\mathrm{X}})$. Given that $\big|\omega\big(\mathrm{d}\varphi(v)\big)\big|\leq\mathrm{Lip}(\varphi)\,|\omega|\,|v|\circ\varphi^{-1}$, we know that $\mathrm{d}\varphi(v)$ is (linear and) continuous. Moreover, for any $f\in L^\infty(\mathfrak{m}_{\mathrm{X}})$ it holds

$$\big(\varphi^*(f\,\omega)(v)\big)\circ\varphi^{-1}=\big(f\circ\varphi\,\varphi^*\omega(v)\big)\circ\varphi^{-1}=f\,\big(\varphi^*\omega(v)\big)\circ\varphi^{-1},$$

thus proving the $L^\infty(\mathfrak{m}_{\mathrm{X}})$-linearity of $\mathrm{d}\varphi(v)$. Hence we have a map $\mathrm{d}\varphi: L^2(T\mathrm{Y})\to L^2(T\mathrm{X})$, which can be easily seen to satisfy all the required properties. □

In the following result, the function $(\gamma,t)\mapsto|\dot\gamma_t|$ is defined everywhere, as in Remark 5.1.

Theorem 22.3 (Speed of a test plan) *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be an infinitesimally Hilbertian metric measure space. Let* $\boldsymbol{\pi}$ *be a test plan on* X. *Then for almost every* $t\in[0,1]$ *there exists an element* $\boldsymbol{\pi}'_t\in\mathrm{e}_t^*L^2(T\mathrm{X})$ *such that*

$$L^1(\boldsymbol{\pi})\text{-}\lim_{h\to 0}\frac{f\circ\mathrm{e}_{t+h}-f\circ\mathrm{e}_t}{h}=[\mathrm{e}_t^*\mathrm{d}f](\boldsymbol{\pi}'_t)\quad \text{for every } f\in W^{1,2}(\mathrm{X}). \tag{22.6}$$

Moreover, the following hold:

i) *the element of* $\mathrm{e}_t^*L^2(T\mathrm{X})$ *satisfying* (22.6) *is unique,*

ii) *we have that* $|\boldsymbol{\pi}'_t|(\gamma)=|\dot\gamma_t|$ *for* $(\boldsymbol{\pi}\times\mathcal{L}_1)$*-a.e.* (γ,t).

Proof. Step 1. Notice that Proposition 19.4 grants that $W^{1,2}(\mathrm{X})$ is separable, thus there exists a countable dense $\mathbb{Q}$-linear subspace D of $W^{1,2}(\mathrm{X})$. By applying Theorem 7.7 we see that for any $f\in D$ it holds that $(f\circ\mathrm{e}_{t+h}-f\circ\mathrm{e}_t)/h$ admits a strong $L^1(\boldsymbol{\pi})$-limit as $h\to 0$ for a.e. t. Moreover, the function $M:[0,1]\to\mathbb{R}$, $M(t):=\int|\dot\gamma_t|^2\,\mathrm{d}\boldsymbol{\pi}(\gamma)$ belongs to $L^1(0,1)$ and the function $(\gamma,t)\mapsto|\dot\gamma_t|$ belongs to $L^2(\boldsymbol{\pi}\times\mathcal{L}_1)$. Hence we can pick a Borel negligible subset $N\subseteq[0,1]$ such that for every $t\in[0,1]\setminus N$ the following hold:

- $\mathsf{Der}_t(f):=\lim_{h\to 0}\big(f\circ\mathrm{e}_{t+h}-f\circ\mathrm{e}_t\big)/h\in L^1(\boldsymbol{\pi})$ exists for every $f\in D$,
- t is a Lebesgue point for M, so that in particular there exists a constant $C_t>0$ with

$$⨍_t^{t+h} M(s)\,\mathrm{d}s\leq C_t\quad \text{for every } h\neq 0 \text{ such that } t+h\in[0,1], \tag{22.7}$$

- the function $\gamma \mapsto |\dot\gamma_t|$ belongs to $L^2(\boldsymbol{\pi})$.

For any $t \in [0,1] \setminus N$, we have that $\mathsf{Der}_t : D \to L^1(\boldsymbol{\pi})$ is a linear operator satisfying (by item ii) of Theorem 7.7) the inequality

$$\big|\mathsf{Der}_t(f)\big|(\gamma) \leq |Df|(\gamma_t)\,|\dot\gamma_t| \quad \text{for } \boldsymbol{\pi}\text{-a.e. } \gamma \tag{22.8}$$

for every $f \in D$. Hence it uniquely extends to a linear continuous $\mathsf{Der}_t : W^{1,2}(\mathrm{X}) \to L^1(\boldsymbol{\pi})$ satisfying the inequality (22.8) for all $f \in W^{1,2}(\mathrm{X})$.

Step 2. Observe that for any $t \in [0,1] \setminus N$ and $g \in W^{1,2}(\mathrm{X})$ it holds that

$$\begin{aligned}\left\|\frac{g \circ \mathrm{e}_{t+h} - g \circ \mathrm{e}_t}{h}\right\|_{L^1(\boldsymbol{\pi})} &\leq \int\!\!\fint_t^{t+h} |Dg|(\gamma_s)\,|\dot\gamma_s|\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\pi}(\gamma)\\ &\leq \left(\int\!\!\fint_t^{t+h} |Dg|^2(\gamma_s)\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\pi}(\gamma)\right)^{1/2}\left(\fint_t^{t+h} M(s)\,\mathrm{d}s\right)^{1/2}\\ &\leq \sqrt{C}\,\big\||Dg|\big\|_{L^2(\mathfrak{m})}\sqrt{C_t}.\end{aligned} \tag{22.9}$$

where C is the compression constant of $\boldsymbol{\pi}$, is satisfied for every $h \neq 0$ such that $t+h \in [0,1]$. Now fix $t \in [0,1] \setminus N$ and $f \in W^{1,2}(\mathrm{X})$. Choose any sequence $(f_n)_n \subseteq D$ that converges to f in $W^{1,2}(\mathrm{X})$. Therefore one has that

$$\begin{aligned}&\left\|\frac{f \circ \mathrm{e}_{t+h} - f \circ \mathrm{e}_t}{h} - \mathsf{Der}_t(f)\right\|_{L^1(\boldsymbol{\pi})}\\ \leq& \sqrt{C\,C_t}\,\big\||D(f-f_n)|\big\|_{L^2(\mathfrak{m})} + \left\|\frac{f_n \circ \mathrm{e}_{t+h} - f_n \circ \mathrm{e}_t}{h} - \mathsf{Der}_t(f_n)\right\|_{L^1(\boldsymbol{\pi})} + \big\|\mathsf{Der}_t(f_n - f)\big\|_{L^1(\boldsymbol{\pi})},\end{aligned}$$

so by first letting $h \to 0$ and then $n \to \infty$ we conclude that $\mathsf{Der}_t(f)$ is the strong $L^1(\boldsymbol{\pi})$-limit of $(f \circ \mathrm{e}_{t+h} - f \circ \mathrm{e}_t)/h$ as $h \to 0$.

Step 3. Call $V_t := \big\{[\mathrm{e}_t^*\mathrm{d}f] \,:\, f \in W^{1,2}(\mathrm{X})\big\}$ for every $t \in [0,1] \setminus N$. Define $L_t : V_t \to L^1(\boldsymbol{\pi})$ as $L_t[\mathrm{e}_t^*\mathrm{d}f] := \mathsf{Der}_t(f)$. Given that for any $f \in W^{1,2}(\mathrm{X})$ property (22.8) yields

$$\big|L_t[\mathrm{e}_t^*\mathrm{d}f]\big|(\gamma) \leq \big|[\mathrm{e}_t^*\mathrm{d}f]\big|(\gamma)\,|\dot\gamma_t| \quad \text{for } \boldsymbol{\pi}\text{-a.e. } \gamma,$$

we see that the operator L_t (is well-defined, linear, continuous and) can be uniquely extended to an element $\boldsymbol{\pi}'_t \in \mathrm{e}_t^*L^2(T\mathrm{X}) \sim \big(\mathrm{e}_t^*L^2(T^*\mathrm{X})\big)^*$. Therefore one has $\mathsf{Der}_t(f) = [\mathrm{e}_t^*\mathrm{d}f](\boldsymbol{\pi}'_t)$ for every $f \in W^{1,2}(\mathrm{X})$ and $|\boldsymbol{\pi}'_t|(\gamma) \leq |\dot\gamma_t|$ for $\boldsymbol{\pi}$-a.e. γ.

Step 4. Given any $f \in \mathrm{LIP}_{bs}(\mathrm{X})$ and $\gamma : [0,1] \to \mathrm{X}$ AC, it holds that $f \circ \gamma$ is AC as well and that for $\boldsymbol{\pi}$-a.e. γ we have $\big(f(\gamma_{t+h}) - f(\gamma_t)\big)/h \to \frac{\mathrm{d}}{\mathrm{d}t}f(\gamma_t)$ as $h \to 0$ for a.e. t. Then

$$[\mathrm{e}_t^*\mathrm{d}f](\boldsymbol{\pi}'_t)(\gamma) = \frac{\mathrm{d}}{\mathrm{d}t}f(\gamma_t) \quad \text{for } (\boldsymbol{\pi} \times \mathcal{L}_1)\text{-a.e. } (\gamma, t).$$

Since $[\mathrm{e}_t^*\mathrm{d}f](\boldsymbol{\pi}'_t)(\gamma) \leq \big|[\mathrm{e}_t^*\mathrm{d}f]\big|(\gamma)\,|\boldsymbol{\pi}'_t|(\gamma) \leq \mathrm{Lip}(f)\,|\boldsymbol{\pi}'_t|(\gamma)$ holds for $\boldsymbol{\pi}$-a.e. γ, we deduce from the previous formula that $\frac{\mathrm{d}}{\mathrm{d}t}f(\gamma_t) \leq \mathrm{Lip}(f)\,|\boldsymbol{\pi}'_t|(\gamma)$ for $\boldsymbol{\pi}$-a.e. γ. In order to conclude, it is thus sufficient to provide the existence of a countable family $D' \subseteq \mathrm{LIP}_{bs}(\mathrm{X})$ of 1-Lipschitz functions such that for every AC curve $\gamma :\, [0,1] \to \mathrm{X}$ it holds

$$|\dot\gamma_t| = \sup_{f \in D'} \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma_t) \qquad \text{for a.e. } t \in [0,1]. \tag{22.10}$$

To do so, fix a countable dense subset $(x_n)_n$ of X and let us define $f_{n,m} := \big(m - \mathsf{d}(\cdot, x_n)\big)^+$ for every $n, m \in \mathbb{N}$. Then the family $D' := (f_{n,m})_{n,m}$ does the job: given any $x, y \in \mathrm{X}$ it clearly holds that $\mathsf{d}(x,y) = \sup_{n,m} \big[f_{n,m}(x) - f_{n,m}(y)\big]$, whence for all $0 \leq s < t \leq 1$ we have

$$\mathsf{d}(\gamma_t, \gamma_s) = \sup_{n,m} \big[f_{n,m}(\gamma_t) - f_{n,m}(\gamma_s)\big] = \sup_{n,m} \int_s^t \frac{\mathrm{d}}{\mathrm{d}r} f_{n,m}(\gamma_r)\,\mathrm{d}r \leq \int_s^t \sup_{n,m} \frac{\mathrm{d}}{\mathrm{d}r} f_{n,m}(\gamma_r)\,\mathrm{d}r.$$

Therefore the thesis is achieved. □

23 Lesson [31/01/2018]

Definition 23.1 (Laplacian) *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an infinitesimally Hilbertian metric measure space. Then a function* $f \in W^{1,2}(\mathrm{X})$ *is in* $D(\Delta)$ *provided there exists* $g \in L^2(\mathfrak{m})$ *such that*

$$\int g\, h \,\mathrm{d}\mathfrak{m} = -\int \nabla f \cdot \nabla h \,\mathrm{d}\mathfrak{m} \qquad \textit{for every } h \in W^{1,2}(\mathrm{X}). \tag{23.1}$$

In this case the function g*, which is uniquely determined by density of* $W^{1,2}(\mathrm{X})$ *in* $L^2(\mathfrak{m})$*, will be denoted by* Δf*.*

Remark 23.2 One has $f \in D(\Delta)$ if and only if $\nabla f \in D(\mathrm{div})$. In this case, $\Delta f = \mathrm{div}(\nabla f)$.

In order to prove it, just observe that

$$\int \mathrm{d}f(\nabla h)\,\mathrm{d}\mathfrak{m} = \int \nabla f \cdot \nabla h \,\mathrm{d}\mathfrak{m} \qquad \text{holds for every } h \in W^{1,2}(\mathrm{X}).$$

In particular, $D(\Delta)$ is a vector space and the map $\Delta :\, D(\Delta) \to L^2(\mathfrak{m})$ is linear. ■

Proposition 23.3 *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be infinitesimally Hilbertian. Then the following hold:*

i) Δ *is a closed operator from* $L^2(\mathfrak{m})$ *to itself,*

ii) *if* $f \in \mathrm{LIP}(\mathrm{X}) \cap D(\Delta)$ *and* $\varphi \in C^2(\mathbb{R})$ *satisfies* $\varphi'' \in L^\infty(\mathbb{R})$*, then* $\varphi \circ f \in D(\Delta)$ *and*

$$\Delta(\varphi \circ f) = \varphi' \circ f\, \Delta f + \varphi'' \circ f\, |\nabla f|^2, \tag{23.2}$$

iii) *if* $f, g \in \mathrm{LIP}_b(\mathrm{X}) \cap D(\Delta)$*, then* $fg \in D(\Delta)$ *and*

$$\Delta(fg) = f\,\Delta g + g\,\Delta f + 2\,\nabla f \cdot \nabla g. \tag{23.3}$$

Proof. i) We aim to show that if $f_n \to f$ and $\Delta f_n \to g$ in $L^2(\mathfrak{m})$, then $f \in D(\Delta)$ and $\Delta f = g$. There exists a constant $C > 0$ such that $\|f_n\|_{L^2(\mathfrak{m})}, \|\Delta f_n\|_{L^2(\mathfrak{m})} \leq C$ for any $n \in \mathbb{N}$, so that

$$\int |\nabla f_n|^2 \,\mathrm{d}\mathfrak{m} \leq -\int f_n \,\Delta f_n \,\mathrm{d}\mathfrak{m} \leq C \quad \text{ for every } n \in \mathbb{N}.$$

This grants that $(f_n)_n$ is bounded in the reflexive space $W^{1,2}(\mathrm{X})$, whence there exists a subsequence $(n_i)_i$ such that $f_{n_i} \rightharpoonup \widetilde{f}$ weakly in $W^{1,2}(\mathrm{X})$, for some $\widetilde{f} \in W^{1,2}(\mathrm{X})$. We already know that $f_{n_i} \to f$ in $L^2(\mathfrak{m})$, then $\widetilde{f} = f$ and accordingly the original sequence $(f_n)_n$ is weakly converging in $W^{1,2}(\mathrm{X})$ to f. Since the differential operator $\mathrm{d} : W^{1,2}(\mathrm{X}) \to L^2(T^*\mathrm{X})$ is linear continuous, we infer that $\mathrm{d}f_n \rightharpoonup \mathrm{d}f$ weakly in $L^2(T^*\mathrm{X})$. By the Riesz isomorphism, this is equivalent to saying that $\nabla f_n \rightharpoonup \nabla f$ weakly in $L^2(T\mathrm{X})$. Therefore

$$-\int h\, g \,\mathrm{d}\mathfrak{m} = -\lim_{n\to\infty} \int h\, \Delta f_n \,\mathrm{d}\mathfrak{m} = \lim_{n\to\infty} \int \nabla f_n \cdot \nabla h \,\mathrm{d}\mathfrak{m} = \int \nabla f \cdot \nabla h \,\mathrm{d}\mathfrak{m}$$

is satisfied for every $h \in W^{1,2}(\mathrm{X})$, thus proving that $f \in D(\Delta)$ and $\Delta f = g$.
ii) Note that $\varphi \circ f \in \mathrm{S}^2(\mathrm{X})$ and $\nabla(\varphi \circ f) = \varphi' \circ f\, \nabla f$. Since $\nabla f \in D(\mathrm{div})$ by Remark 23.2 and $\varphi' \circ f \in \mathrm{LIP}_b(\mathrm{X})$, we deduce from Proposition 17.10 that $\nabla(\varphi \circ f) \in D(\mathrm{div})$ and

$$\Delta(\varphi \circ f) = \mathrm{div}\big(\varphi' \circ f\, \nabla f\big) = \mathrm{d}(\varphi' \circ f)(\nabla f) + \varphi' \circ f\, \mathrm{div}(\nabla f) = \varphi'' \circ f\, |\nabla f|^2 + \varphi' \circ f\, \Delta f,$$

which proves (23.2).
iii) Note that $fg \in \mathrm{S}^2(\mathrm{X})$ and $\nabla(fg) = f\, \nabla g + g\, \nabla f$. By applying again Proposition 17.10, we deduce that $\nabla(fg) \in D(\mathrm{div})$ and

$$\begin{aligned} \Delta(fg) &= \mathrm{div}\big(f\, \nabla g + g\, \nabla f\big) = \mathrm{d}f(\nabla g) + f\, \mathrm{div}(\nabla g) + \mathrm{d}g(\nabla f) + g\, \mathrm{div}(\nabla f) \\ &= f\, \Delta g + g\, \Delta f + 2\, \nabla f \cdot \nabla g, \end{aligned}$$

which proves (23.3). □

We now provide an alternative characterisation of the Laplacian operator.
Let H be a Hilbert space and let $E : H \to [0, +\infty]$ be a convex lsc functional. Given any point $x \in H$ such that $E(x) < \infty$, we define the *subdifferential* of E at x as

$$\partial^- E(x) := \big\{ v \in H \ : \ E(x) + \langle v, y - x \rangle \leq E(y) \text{ for every } y \in H \big\}. \tag{23.4}$$

It trivially holds that $0 \in \partial^- E(x)$ if and only if x is a minimum point of E.

Exercise 23.4 Consider $H := \mathbb{R}$ and $E(x) := |x|$ for every $x \in \mathbb{R}$. Then

$$\partial^- E(x) := \begin{cases} \{1\} & \text{if } x > 0, \\ [-1, 1] & \text{if } x = 0, \\ \{-1\} & \text{if } x < 0. \end{cases} \tag{23.5}$$

Proposition 23.5 *The following hold:*

i) *the multivalued map* $\partial^- E : H \to 2^H$ *is a* monotone operator*, i.e.*

$$\langle x - y, v - w \rangle \geq 0 \qquad \textit{for every } x, y \in H,\ v \in \partial^- E(x) \textit{ and } w \in \partial^- E(y), \tag{23.6}$$

ii) *the set* $\{(x, v) \in H \times H : v \in \partial^- E(x)\}$ *is closed in* $H \times H$.

Proof. i) From $v \in \partial^- E(x)$ and $w \in \partial^- E(y)$ we deduce that

$$\begin{aligned} E(x) + \langle v, y - x \rangle &\leq E(y), \\ E(y) + \langle w, x - y \rangle &\leq E(x), \end{aligned} \tag{23.7}$$

respectively. By summing the two in (23.7) we obtain $\langle v - w, y - x \rangle \leq 0$, proving (23.6).
ii) Fix two sequences $(x_n)_n, (v_n)_n \subseteq H$ such that $x_n \to x$, $v_n \to v$ and $v_n \in \partial^- E(x_n)$. Hence for any $y \in H$ it holds that

$$E(x) + \langle v, y - x \rangle \leq \varliminf_{n \to \infty} E(x_n) + \lim_{n \to \infty} \langle v_n, y - x_n \rangle \leq E(y),$$

thus showing that $v \in \partial^- E(x)$. This proves the statement. □

Remark 23.6 It actually holds that $\{(x, v) \in H \times H : v \in \partial^- E(x)\}$ is *strongly-weakly closed* in $H \times H$, which means that

$$\left.\begin{aligned} x_n \to x &\text{ strongly in } H, \\ v_n \rightharpoonup v &\text{ weakly in } H, \\ v_n \in \partial^- E(x_n) &\text{ for all } n \end{aligned}\right\} \quad \Longrightarrow \quad v \in \partial^- E(x). \tag{23.8}$$

The proof is the same of item ii) of Proposition 23.5. ■

Proposition 23.7 *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be infinitesimally Hilbertian. Call* $E : L^2(\mathfrak{m}) \to [0, +\infty]$ *the Cheeger energy, which is the convex lsc functional that is defined as* $E(f) := \frac{1}{2} \int |\nabla f|^2 \, d\mathfrak{m}$ *for any* $f \in W^{1,2}(\mathrm{X})$ *and equal to* $+\infty$ *elsewhere. Then a function* $f \in W^{1,2}(\mathrm{X})$ *belongs to* $D(\Delta)$ *if and only if* $\partial^- E(f) \neq \emptyset$*. In this case, it holds that* $\partial^- E(f) = \{-\Delta f\}$*.*

Proof. First of all, observe that for any $f, g \in W^{1,2}(\mathrm{X})$ we have that

$$\mathbb{R} \ni \varepsilon \mapsto E(f + \varepsilon g) \text{ is convex and } \lim_{\varepsilon \to 0} \frac{E(f + \varepsilon g) - E(f)}{\varepsilon} = \int \nabla f \cdot \nabla g \, d\mathfrak{m}, \tag{23.9}$$

as one can readily deduce from the fact that $E(f + \varepsilon g) = \frac{1}{2} \int |\nabla f|^2 + 2 \varepsilon \nabla f \cdot \nabla g + \varepsilon^2 |\nabla g|^2 \, d\mathfrak{m}$.

Let $f \in D(\Delta)$. We want to show that $E(f) - \int g \, \Delta f \, d\mathfrak{m} \leq E(f + g)$ for every $g \in W^{1,2}(\mathrm{X})$. In order to prove it, just notice that (23.9) yields

$$E(f + g) - E(f) \geq \lim_{\varepsilon \searrow 0} \frac{E(f + \varepsilon g) - E(f)}{\varepsilon} = \int \nabla f \cdot \nabla g \, d\mathfrak{m} = -\int g \, \Delta f \, d\mathfrak{m},$$

which grants that $-\Delta f \in \partial^- E(f)$.

Conversely, let $v \in \partial^- E(f)$. Then $\varepsilon \int v \, g \, d\mathfrak{m} \leq E(f + \varepsilon g) - E(f)$ holds for every $\varepsilon \in \mathbb{R}$ and $g \in W^{1,2}(\mathrm{X})$. Therefore we have that

$$\int \nabla f \cdot \nabla g \, d\mathfrak{m} = \lim_{\varepsilon \searrow 0} \frac{E(f - \varepsilon g) - E(f)}{-\varepsilon} \leq \int v \, g \, d\mathfrak{m} \leq \lim_{\varepsilon \searrow 0} \frac{E(f + \varepsilon g) - E(f)}{\varepsilon} = \int \nabla f \cdot \nabla g \, d\mathfrak{m}$$

for every $g \in W^{1,2}(\mathrm{X})$. This says that $f \in D(\Delta)$ and $\Delta f = -v$. □

Lemma 23.8 *Let H be a Hilbert space. Let $[0,1] \ni t \mapsto v_t \in H$ be an AC curve. Then*

$$\exists \lim_{h \to 0} \frac{v_{t+h} - v_t}{h} =: v'_t \in H \quad \text{for a.e. } t \in [0,1]. \tag{23.10}$$

Moreover, the map $t \mapsto v'_t$ belongs to $L^1\big([0,1], H\big)$ and satisfies

$$v_t - v_s = \int_s^t v'_r \, \mathrm{d}r \quad \text{for every } s, t \in [0,1] \text{ with } s < t. \tag{23.11}$$

Proof. Since v is essentially separably valued, we assume with no loss of generality that H is separable. Fix an orthonormal basis $(\mathsf{e}_n)_n$ of H. Given any $n \in \mathbb{N}$, we have that $t \mapsto v_t \cdot \mathsf{e}_n \in \mathbb{R}$ is AC and accordingly a.e. differentiable. Hence there exists a Borel negligible set $N \subseteq [0,1]$ such that

$$\exists\, \ell_n(t) := \lim_{h \to 0} \frac{v_{t+h} \cdot \mathsf{e}_n - v_t \cdot \mathsf{e}_n}{h} \in \mathbb{R} \quad \text{for every } n \in \mathbb{N} \text{ and } t \in [0,1] \setminus N.$$

For any $k \in \mathbb{N}$, call $L_k(t) := \sum_{n=0}^k \ell_n(t)\, \mathsf{e}_n \in H$ if $t \in [0,1] \setminus N$ and $L_k(t) := 0 \in H$ if $t \in N$. Clearly the map $L_k : [0,1] \to H$ is strongly Borel. Moreover, for any $k \in \mathbb{N}$ it holds that

$$\begin{aligned} \sum_{n=0}^{\infty} \big|\ell_n(t)\big|^2 &= \lim_{k \to \infty} \sum_{n=0}^{k} \big|\ell_n(t)\big|^2 = \lim_{k \to \infty} \lim_{h \to 0} \sum_{n=0}^{k} \left| \frac{v_{t+h} - v_t}{h} \cdot \mathsf{e}_n \right|^2 \\ &\leq \lim_{h \to 0} \left\| \frac{v_{t+h} - v_t}{h} \right\|_H^2 = |\dot{v}_t|^2 < +\infty \qquad \text{for a.e. } t \in [0,1] \setminus N. \end{aligned} \tag{23.12}$$

In particular, for a.e. $t \in [0,1] \setminus N$ there exists $L(t) \in H$ such that $\lim_k \big\|L_k(t) - L(t)\big\|_H = 0$. We also deduce from (23.12) that $\big\|L(t)\big\|_H \leq |\dot{v}_t|$ for a.e. $t \in [0,1]$, whence $L : [0,1] \to H$ is Bochner integrable by Proposition 5.13. By applying the dominated convergence theorem, we see that $\int_s^t L(r)\, \mathrm{d}r = \lim_k \int_s^t L_k(r)\, \mathrm{d}r$ for every $t, s \in [0,1]$ with $s \leq t$, so that

$$\begin{aligned} v_t - v_s &= \lim_{k \to \infty} \sum_{n=0}^{k} \big[(v_t - v_s) \cdot \mathsf{e}_n\big] \mathsf{e}_n = \lim_{k \to \infty} \sum_{n=0}^{k} \left(\int_s^t \ell_n(r)\, \mathrm{d}r \right) \mathsf{e}_n \overset{(6.3)}{=} \lim_{k \to \infty} \int_s^t L_k(r)\, \mathrm{d}r \\ &= \int_s^t L(r)\, \mathrm{d}r. \end{aligned}$$

Hence v is a.e. differentiable, with derivative $v' := L$, proving the statement. □

Theorem 23.9 (Heat flow) *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be an infinitesimally Hilbertian metric measure space. Then for every $f \in L^2(\mathfrak{m})$ there exists a unique map $[0, +\infty) \ni t \mapsto f_t \in L^2(\mathfrak{m})$ with the following properties:*

i) *$f_0 = f$ and $[0, +\infty) \ni t \mapsto f_t \in L^2(\mathfrak{m})$ is continuous,*

ii) *the map $(0, +\infty) \ni t \mapsto f_t \in L^2(\mathfrak{m})$ is locally absolutely continuous,*

iii) *for a.e. $t > 0$ it holds that $f_t \in D(\Delta)$ and $f'_t = \Delta f_t$.*

The previous theorem is a special case of the following result:

Theorem 23.10 (Gradient flow) *Let H be a Hilbert space and let $E : H \to [0,+\infty]$ be a convex lsc functional whose domain $D(E) := \big\{v \in H \,:\, E(v) < +\infty\big\}$ is dense in H. Then for every $v \in H$ there exists a unique map $[0,+\infty) \ni t \mapsto v_t \in H$ with the following properties:*

i) *$v_0 = v$ and $[0,+\infty) \ni t \mapsto v_t \in H$ is continuous,*

ii) *the map $(0,+\infty) \ni t \mapsto v_t \in H$ is locally absolutely continuous,*

iii) *for a.e. $t > 0$ it holds that $-v_t' \in \partial^- E(v_t)$.*

24 Lesson [05/02/2018]

Let H be a Hilbert space. Let $E : H \to [0,+\infty]$ be a convex lsc functional that is not identically equal to $+\infty$. Then we define

$$\begin{aligned} D(E) &:= \big\{x \in H \,:\, E(x) < +\infty\big\}, \\ D(\partial^- E) &:= \big\{x \in H \,:\, \partial^- E(x) \neq \emptyset\big\} \subseteq D(E). \end{aligned}$$

The *slope* of E is the functional $|\partial^- E| : H \to [0,+\infty]$ given by

$$|\partial^- E|(x) := \begin{cases} \sup_{y \neq x} \big(E(y) - E(x)\big)^- / |x - y| & \text{if } x \in D(E), \\ +\infty & \text{otherwise.} \end{cases}$$

Observe that $|\partial^- E|(x) = 0$ if and only if x is a minimum point of E.

Remark 24.1 We claim that

$$|\partial^- E|(x) \leq |v| \quad \text{ for every } v \in \partial^- E(x). \tag{24.1}$$

Indeed, we know that $E(x) + \langle v, y - x\rangle \leq E(y)$ for any $y \in H$, so that $E(x) - E(y) \leq |v|\,|x-y|$ and accordingly $\big(E(x) - E(y)\big)^+ \leq |v|\,|x-y|$ for any $y \in H$, which gives (24.1). ■

Exercise 24.2 Given any $x \in H$ and $\tau > 0$, let us define

$$F_{x,\tau}(\cdot) := E(\cdot) + \frac{|\cdot - x|^2}{2\,\tau}. \tag{24.2}$$

Then it holds that $\partial^- F_{x,\tau}(y) = \partial^- E(y) + \frac{y-x}{\tau}$ for every $y \in H$.

Proposition 24.3 *Let $x \in H$ and $\tau > 0$. Then there exists a unique minimiser $x_\tau \in H$ of the functional $F_{x,\tau}$ defined in (24.2). Moreover, it holds that $\frac{x_\tau - x}{\tau} \in -\partial^- E(x_\tau)$.*

Proof. Since E is convex lsc and $|\cdot - x|^2/(2\,\tau)$ is strictly convex and continuous, we get that the functional $F_{x,\tau}$ is strictly convex and lsc. This grants that the sublevels of $F_{x,\tau}$ are convex and strongly closed, so that they are also weakly closed by Hahn-Banach theorem, in other words $F_{x,\tau}$ is weakly lsc. Moreover, the sublevels of $|\cdot - x|^2/(2\,\tau)$ are bounded, whence those of $F_{x,\tau}$ are bounded as well, thus is particular they are weakly compact. Then Bolzano-Weierstrass theorem yields existence of a minimum point $x_\tau \in H$ of $F_{x,\tau}$, which is unique by strict convexity of $F_{x,\tau}$. Finally, since the point x_τ is a minimiser for $F_{x,\tau}$, we know from Exercise 24.2 that $0 \in \partial^- F_{x,\tau}(x_\tau) = \partial^- E(x_\tau) + \frac{x_\tau - x}{\tau}$, or equivalently $\frac{x_\tau - x}{\tau} \in -\partial^- E(x_\tau)$, which gives the last statement. □

Corollary 24.4 *It holds that $D(\partial^- E)$ is dense in $D(E)$ and that*

$$|\partial^- E|(x_\tau) \leq \frac{|x_\tau - x|}{\tau} \leq |\partial^- E|(x) \quad \textit{for every } x \in H \textit{ and } \tau > 0. \tag{24.3}$$

Proof. Given any $x \in D(E)$, we deduce from the very definition of x_τ that

$$\varlimsup_{\tau \searrow 0} |x_\tau - x|^2 \leq \varlimsup_{\tau \searrow 0} 2\,\tau \left(E(x_\tau) + \frac{|x_\tau - x|^2}{2\,\tau} \right) \leq \lim_{\tau \searrow 0} 2\,\tau\, E(x) = 0,$$

whence the first statement follows. Moreover, since $\frac{x - x_\tau}{\tau} \in \partial^- E(x_\tau)$ by Proposition 24.3, we infer from (24.1) that $|x_\tau - x|/\tau \geq |\partial^- E|(x_\tau)$. To conclude, define $z_\lambda := (1-\lambda)\,x + \lambda\, x_\tau$ for every $\lambda \in [0,1]$. The minimality of x_τ and the convexity of E give

$$E(x_\tau) + \frac{|x_\tau - x|^2}{2\,\tau} \leq E(z_\lambda) + \frac{|z_\lambda - x|^2}{2\,\tau} \leq (1-\lambda)\,E(x) + \lambda\,E(x_\tau) + \lambda^2\,\frac{|x_\tau - x|^2}{2\,\tau}$$

for every $\lambda \in [0,1]$, which can be rewritten as

$$(1-\lambda)\big(E(x) - E(x_\tau)\big) \geq (1-\lambda^2)\,\frac{|x_\tau - x|^2}{2\,\tau} \qquad \text{for every } \lambda \in [0,1],$$

so that $\frac{E(x)-E(x_\tau)}{|x_\tau - x|} \geq (1+\lambda)\,\frac{|x_\tau - x|}{2\,\tau}$ for all $\lambda \in [0,1]$. By letting $\lambda \nearrow 1$ in such inequality, we conclude that $|\partial^- E|(x) \geq \frac{E(x)-E(x_\tau)}{|x_\tau - x|} \geq \frac{|x_\tau - x|}{\tau}$. Hence the thesis is achieved. □

Remark 24.5 We claim that the functional $|\partial^- E| : H \to [0,+\infty]$ is lsc.
In order to prove it, for any $y \in H$ we define $G_y : H \to [0,+\infty]$ as

$$G_y(x) := \begin{cases} \big(E(y) - E(x)\big)^- / |x - y| & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases}$$

with the convention that $\big(E(y) - E(x)\big)^- := +\infty$ when $E(x) = E(y) = +\infty$. It can be readily checked that $|\partial^- E|(x) = \sup_{y \in H} G_y(x)$ for every $x \in H$. Given that each functional G_y is lsc by construction, we conclude that $|\partial^- E|$ is lsc as well. ■

Lemma 24.6 *It holds that*

$$|\partial^- E|(x) = \min_{v \in \partial^- E(x)} |v| \quad \textit{for every } x \in H. \tag{24.4}$$

Proof. The inequality $\leq$ is granted by (24.1). To prove $\geq$, notice that $|\partial^- E|(x) \geq |x - x_\tau|/\tau$ for all $\tau > 0$ by (24.3). We can clearly assume wlog that $x \in D(\partial^- E)$. Therefore there exists a sequence $(\tau_n)_n \searrow 0$ such that $\frac{x - x_{\tau_n}}{\tau_n} \rightharpoonup v$ weakly in H as $n \to \infty$, for some $v \in H$. Since we have that $\frac{x - x_{\tau_n}}{\tau_n} \in \partial^- E(x_{\tau_n})$ for all $n \in \mathbb{N}$, we infer from Remark 23.6 that $v \in \partial^- E(x)$. Given that $|v| \leq \varliminf_n |x_{\tau_n} - x|/\tau_n \leq |\partial^- E|(x)$, we proved the statement. □

Remark 24.7 It is clear that the set $\partial^- E(x)$ is closed and convex for every $x \in H$. In particular, if $x \in D(\partial^- E)$ then $\partial^- E(x)$ admits a unique element of minimal norm. ■

We now restate Theorem 23.10 (with some additional statements) and prove it.

Theorem 24.8 (Gradient flow) *Let $x \in \overline{D(E)}$ be fixed. Then there exists a unique continuous curve $[0, +\infty) \ni t \mapsto x_t \in H$ starting from x, called* gradient flow*, which is locally AC on $(0, +\infty)$ and satisfies $x'_t \in -\partial^- E(x_t)$ for a.e. $t \in [0, +\infty)$. Moreover, the following hold:*

1) (Contraction property) *Given two gradient flows (x_t) and (y_t), we have that*

$$|x_t - y_t| \le |x_0 - y_0| \quad \text{for every } t \ge 0. \tag{24.5}$$

2) *The maps $t \mapsto x_t$ and $t \mapsto E(x_t)$ are locally Lipschitz on $(0, +\infty)$.*

3) *The functions $t \mapsto E(x_t)$ and $t \mapsto |\partial^- E|(x_t)$ are non increasing on $[0, +\infty)$.*

4) *For any $y \in H$, we have that $E(x_t) + \langle x'_t, x_t - y\rangle \le E(y)$ holds for a.e. $t \in (0, +\infty)$.*

5) *We have that $-\frac{\mathrm{d}}{\mathrm{d}t} E(x_t) = |\dot{x}_t|^2 = |\partial^- E|^2(x_t)$ for a.e. $t \in [0, +\infty)$.*

6) *The following inequalities are satisfied:*

   6a) *$E(x_t) \le E(y) + \frac{|x_0 - y|^2}{2\,t}$ for every $y \in H$ and $t \ge 0$.*

   6b) *$|\partial^- E|^2(x_t) \le |\partial^- E|^2(y) + \frac{|x_0 - y|^2}{t^2}$ for every $y \in H$ and $t \ge 0$.*

7) *For any $t > 0$, we have that the incremental ratio $\frac{x_{t+h} - x_t}{h}$ converges to the element of minimal norm of $\partial^- E(x_t)$ as $h \searrow 0$. The same holds for $t = 0$ provided $\partial^- E(x_0) \neq \emptyset$.*

Proof. Step 1. We start by proving existence in the case $x \in D(E)$. Fix $\tau > 0$. We recursively define the sequence $(x^\tau_{(n)})_n \subseteq H$ as $x^\tau_{(0)} := x$ and

$$x^\tau_{(n+1)} := \operatorname*{argmin}_H \left(E(\cdot) + \frac{|\cdot - x^\tau_{(n)}|^2}{2\,\tau} \right) \qquad \text{for every } n \in \mathbb{N}.$$

Then define (x^τ_t) as the unique curve in H such that $x^\tau_{n\tau} = x^\tau_{(n)}$ for all $n \in \mathbb{N}$ and that is affine on each interval $[n\tau, (n+1)\tau]$. For any $n \in \mathbb{N}$, we clearly have that

$$(x^\tau_t)' = \frac{x^\tau_{(n+1)} - x^\tau_{(n)}}{\tau} \qquad \text{for every } t \in (n\tau, (n+1)\tau). \tag{24.6}$$

Since $E(x^\tau_{(n+1)}) + |x^\tau_{(n+1)} - x^\tau_{(n)}|^2/(2\,\tau) \le E(x^\tau_{(n)})$ for all $n \in \mathbb{N}$, we infer from (24.6) that

$$\frac{1}{2} \int_0^{+\infty} |\dot{x}^\tau_t|^2 \, \mathrm{d}t = \sum_{n=0}^{\infty} \frac{|x^\tau_{(n+1)} - x^\tau_{(n)}|^2}{2\,\tau} \le E(x) < +\infty. \tag{24.7}$$

Given $\tau, \eta > 0$ and $k, k' \in \mathbb{N}$ such that $t \in ((k-1)\tau, k\tau] \cap ((k'-1)\eta, k'\eta]$, it holds that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \frac{|x_t^\tau - x_t^\eta|^2}{2} &= \underbrace{\big\langle (x_t^\tau)' - (x_t^\eta)', x_{k\tau}^\tau - x_{k'\eta}^\eta \big\rangle}_{\leq 0 \text{ by } (23.6)} + \big\langle (x_t^\tau)' - (x_t^\eta)', (x_t^\tau - x_{k\tau}^\tau) - (x_t^\eta - x_{k'\eta}^\eta) \big\rangle \\
&\leq \big(|(x_t^\tau)'| + |(x_t^\eta)'| \big) \big(\tau \, |(x_t^\tau)'| + \eta \, |(x_t^\eta)'| \big) \\
&\leq |(x_t^\tau)'|^2 \left(\tau + \frac{\tau + \eta}{2} \right) + |(x_t^\eta)'|^2 \left(\eta + \frac{\tau + \eta}{2} \right).
\end{aligned}
$$

By integrating over the interval $[0, T]$, we thus deduce from (24.7) that

$$
\frac{|x_T^\tau - x_T^\eta|^2}{2} \leq 2 \, E(x) \, (\tau + \eta) \qquad \text{for every } \tau, \eta > 0. \tag{24.8}
$$

This grants that $\sup_{t \geq 0} |x_t^\tau - x_t^\eta| \to 0$ as $\tau, \eta \searrow 0$, so there exists a continuous curve (x_t), with $x_0 = x$, which is the uniform limit of (x_t^τ) as $\tau \searrow 0$.

Notice that $\big\{ (x_\cdot^\tau)' \in L^2\big([0, +\infty), H\big) \;\big|\; \tau > 0 \big\}$ is norm bounded by (24.7), so that there exists $(\tau_n)_n \searrow 0$ such that $(x_\cdot^{\tau_n})' \rightharpoonup v_\cdot$ weakly in $L^2\big([0, +\infty), H\big)$ as $n \to \infty$, for a suitable limit $v_\cdot \in L^2\big([0, +\infty), H\big)$. Given any $\varphi \in C_c^\infty(0, +\infty)$, one has that

$$
\int_0^{+\infty} \varphi_t \, (x_t^{\tau_n})' \, \mathrm{d}t = - \int_0^{+\infty} \varphi_t' \, x_t^{\tau_n} \, \mathrm{d}t \qquad \text{for every } n \in \mathbb{N},
$$

thus by letting $n \to \infty$ we get that $\int_0^{+\infty} \varphi_t \, v_t \, \mathrm{d}t = - \int_0^{+\infty} \varphi_t' \, x_t \, \mathrm{d}t$. This ensures that the curve (x_t) is locally AC on $(0, +\infty)$. Now let $y \in H$ be fixed. We claim that

$$
\int_{t_0}^{t_1} E(x_t) + \langle x_t', x_t - y \rangle \, \mathrm{d}t \leq (t_1 - t_0) \, E(y) \qquad \text{for every } 0 \leq t_0 \leq t_1 < +\infty. \tag{24.9}
$$

Recall that $-(x_{(n+1)}^\tau - x_{(n)}^\tau)/\tau \in \partial^- E(x_{(n+1)}^\tau)$ for all $n \in \mathbb{N}$. Moreover, it holds that

$$
\int_0^\tau E(x_t^\tau) \, \mathrm{d}t \leq \int_0^\tau \left(1 - \frac{t}{\tau} \right) E(x_0) + \frac{t}{\tau} \, E(x_{(1)}^\tau) \, \mathrm{d}t = \frac{\tau}{2} \, E(x_0) + \frac{\tau}{2} \, E(x_{(1)}^\tau).
$$

Therefore simple computations yield

$$
\begin{aligned}
\int_{t_0}^{t_1} E(x_t) + \langle x_t', x_t - y \rangle \, \mathrm{d}t &\leq \varliminf_{\tau \searrow 0} \int_{t_0}^{t_1} E(x_t^\tau) + \big\langle (x_t^\tau)', x_t^\tau - y \big\rangle \, \mathrm{d}t \\
&\leq \varliminf_{\tau \searrow 0} \int_{t_0}^{t_1} E(x_{[t/\tau+1]\tau}^\tau) + \big\langle (x_t^\tau)', x_{[t/\tau+1]\tau}^\tau - y \big\rangle \, \mathrm{d}t \\
&\leq \varliminf_{\tau \searrow 0} \int_{t_0}^{t_1} E(y) \, \mathrm{d}t = (t_1 - t_0) \, E(y),
\end{aligned}
$$

which proves the validity of our claim (24.9). Finally, take $t > 0$ that is both a Lebesgue point for $E(x_\cdot)$ and a differentiability point for $x_\cdot$ (almost every $t > 0$ has this property). Then it follows from (24.9) that the formula in item 4) is verified at such t, proving that (x_t) is a gradient flow starting from x. Hence existence and item 4) are proven for $x \in D(E)$.

STEP 2. Suppose that (x_t), (y_t) are gradient flows starting from points in $\overline{D(E)}$. Then the function $t \mapsto \frac{|x_t-y_t|^2}{2}$ is continuous on $[0,+\infty)$ and locally AC on $(0,+\infty)$. Item i) of Proposition 23.5 yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{|x_t-y_t|^2}{2} = \langle x_t' - y_t', x_t - y_t\rangle \leq 0 \qquad \text{for a.e. } t>0.$$

Hence $|x_t - y_t| \leq |x_0 - y_0|$ for every $t \geq 0$, proving i) and uniqueness of the gradient flow.
STEP 3. We aim to prove existence and uniqueness of the gradient flow starting from some point $x \in \overline{D(E)}$. Choose any sequence $(x^n)_n \subseteq D(E)$ such that $x^n \to x$. Call (x^n_t) the gradient flow with initial datum x^n. We know from the contraction property 1) that

$$\sup_{t\geq 0} |x^n_t - x^m_t| \leq |x^n - x^m| \to 0 \qquad \text{as } n,m\to\infty,$$

so there is a continuous curve (x_t) that is uniform limit of (x^n_t). Given $y \in D(E)$ and $t_0 > 0$, there exists a constant $C(t_0) > 0$ such that

$$E(x^n_{t_0}) \leq E(y) + \frac{|x_n - y|^2}{2\,t_0} \leq C(t_0) \qquad \text{for every } n \in \mathbb{N},$$

whence from (24.7) it follows that $\frac{1}{2}\int_0^{+\infty} |\dot{x}^n_t|^2 \,\mathrm{d}t \leq C(t_0)$ holds for every $n \in \mathbb{N}$. In other words, $(x^n_\cdot)$ are uniformly AC on $[t_0,+\infty)$. Hence $(x^n_\cdot)' \rightharpoonup x'_\cdot$ weakly in $L^2\big([t_0,+\infty),H\big)$, which is enough to conclude.
STEP 4. We aim to prove 3). Fix $0 \leq t_0 \leq t_1 < +\infty$. Call (x_t) the gradient flow starting from some point $x \in \overline{D(E)}$, then (y_t) the gradient flow starting from x_{t_0}. By uniqueness, we have that $x_{t_1} = y_{t_1 - t_0}$. Furthermore, one has $E(x_{t_1}) = E(y_{t_1-t_0}) \leq E(y_0) = E(x_{t_0})$ by construction. This shows that $t \mapsto E(x_t)$ is a non increasing function. A similar argument based on (24.3) grants that $t \mapsto |\partial^- E|(x_t)$ is non increasing as well. Then iii) is proven.
STEP 5. We want to prove 6a). Fix $x \in \overline{D(E)}$ and call (x_t) the gradient flow with $x_0 = x$. Let $y \in H$ and $t \geq 0$. By integrating the inequality in 4) on $[0,t]$ and by recalling 3), we get

$$t\,E(x_t) \leq \int_0^t E(x_s)\,\mathrm{d}s \leq t\,E(y) - \frac{|x_t-y|^2}{2} + \frac{|x-y|^2}{2} \leq t\,E(y) + \frac{|x-y|^2}{2},$$

whence 6a) immediately follows.

(To be continued in the next lesson...) □

25 Lesson [07/02/2018]

We conclude the proof of Theorem 24.8.

Proof. STEP 6. Fix $\varepsilon > 0$. Since the curve (x_t) is locally AC on $(0,\varepsilon)$, there exists $t_0 \in (0,\varepsilon)$ such that x'_{t_0} exists. Moreover, for any $s \geq 0$ it holds that $t \mapsto x_{t+s}$ is the gradient flow starting from x_s. Therefore we have that

$$|\dot{x}_{t_0+s}| = \lim_{t\searrow t_0} \frac{|x_{t+s} - x_{t_0+s}|}{|t-t_0|} \overset{1)}{\leq} \lim_{t\searrow t_0}\frac{|x_t - x_{t_0}|}{|t-t_0|} = |\dot{x}_{t_0}| \qquad \text{holds for a.e. } s\geq 0,$$

which grants that the metric speed $|\dot{x}|$ is bounded in $[\varepsilon, \infty)$. This means that (x_t) is Lipschitz on $[\varepsilon, +\infty)$. Now call L_ε its Lipschitz constant. Item 4) ensures that for any $y \in H$ one has

$$E(x_t) - L_\varepsilon\,|x_t - x| \le E(x_t) - |\dot{x}_t|\,|x_t - y| \le E(x_t) - \langle x'_t, x_t - y\rangle \le E(y)$$

for a.e. $t \in (\varepsilon, +\infty)$, thus also for every $t > \varepsilon$ by lsc of E. By choosing $y = x_s$, we see that the inequality $E(x_t) - E(x_s) \le L_\varepsilon |x_t - x_s|$ holds for all $s, t > \varepsilon$. This shows that $t \mapsto E(x_t)$ is Lipschitz, thus concluding the proof of 2).

Step 7. We now prove item 5). Since $\frac{E(x_t)-E(y)}{|x_t-y|} \le |\dot{x}_t|$ holds for every $y \in H$ and a.e. t by property 4), we deduce that

$$|\partial^- E|(x_t) = \sup_{y \ne x_t} \frac{\big(E(x_t) - E(y)\big)^+}{|x_t - y|} \le |\dot{x}_t| \quad \text{for a.e. } t \ge 0. \tag{25.1}$$

Moreover, observe that for a.e. $t \ge 0$ it holds that

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t} E(x_t) &= \lim_{h\to 0} \frac{E(x_t) - E(x_{t+h})}{h} \le |\partial^- E|(x_t) \lim_{h\to 0} \frac{|x_{t+h} - x_t|}{|h|} = |\partial^- E|(x_t)\,|\dot{x}_t| \\ &\le \frac{1}{2}\,|\partial^- E|^2(x_t) + \frac{1}{2}\,|\dot{x}_t|^2. \end{aligned} \tag{25.2}$$

Recall that $E(x_s) - |\dot{x}_s|\,|x_s - y| \le E(y)$ holds for every $y \in H$ and a.e. $s > t > 0$. By integrating it on the interval $[t, t+h]$, we thus obtain that

$$\frac{|x_{t+h} - y|^2}{2} - \frac{|x_t - y|^2}{2} + \int_t^{t+h} E(x_s)\,\mathrm{d}s \le h\,E(y) \quad \text{for every } y \in H \text{ and } t, h \ge 0.$$

By using the previous inequality with $y = x_t$ and the dominated convergence theorem, we get

$$\begin{aligned} \frac{|\dot{x}_t|^2}{2} &= \lim_{h\searrow 0} \frac{|x_{t+h} - x_t|^2}{2\,h^2} \le \lim_{h\searrow 0} ⨍_t^{t+h} \frac{E(x_t) - E(x_s)}{h}\,\mathrm{d}s \\ &= \lim_{h\searrow 0} \int_0^1 \frac{E(x_t) - E(x_{t+hr})}{h\,r}\,r\,\mathrm{d}r = -\frac{\mathrm{d}}{\mathrm{d}t} E(x_t) \int_0^1 r\,\mathrm{d}r \\ &= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} E(x_t) \quad \text{for a.e. } t > 0. \end{aligned} \tag{25.3}$$

Finally, we obtain 5) by putting together (25.1), (25.2) and (25.3).

Step 8. We want to prove 6b). Since the slope $|\partial^- E|$ is lsc (cf. Remark 24.5), it suffices to prove it for $x_0 \in D(E)$. Notice that the Young inequality yields

$$t\big(E(y) - E(x_t)\big) \le t\,|\partial^- E|(y)\,|y - x_t| \le \frac{t^2\,|\partial^- E|^2(y)}{2} + \frac{|x_t - y|^2}{2}. \tag{25.4}$$

By using (25.4) and items 3), 4), 5), we see that

$$\begin{aligned} \frac{t^2\,|\partial^- E|^2(x_t)}{2} &\le \int_0^t s\,|\partial^- E|^2(x_s)\,\mathrm{d}s = -\int_0^t s\,\frac{\mathrm{d}}{\mathrm{d}s} E(x_s)\,\mathrm{d}s = \int_0^t E(x_s)\,\mathrm{d}s - t\,E(x_t) \\ &\le t\,E(y) + \frac{|x_0 - y|^2}{2} - \frac{|x_t - y|^2}{2} - t\,E(x_t) \le \frac{t^2\,|\partial^- E|^2(y)}{2} + \frac{|x_0 - y|^2}{2}, \end{aligned}$$

which proves 6b).

STEP 9. It only remains to prove 7). It is enough to prove it for $t=0$ and $|\partial^- E|(x_0)<+\infty$. Observe that $\left|\frac{x_h-x}{h}\right| \leq \fint_0^h |\dot{x}_t|\,\mathrm{d}t \leq |\partial^- E|(x_0)$ for all $h>0$ by 3) and 5). Hence there exists a sequence $(h_n)_n \searrow 0$ such that $\frac{x_{h_n}-x_0}{h_n} \rightharpoonup v \in H$. Clearly $|v| \leq |\partial^- E|(x_0)$. By recalling Lemma 24.6, we thus see that it just remains to show that $v \in \partial^- E(x_0)$. Notice that

$$\fint_0^{h_n} \langle x_t', x_t - y\rangle\,\mathrm{d}t = \Big\langle \fint_0^{h_n} x_t'\,\mathrm{d}t, x_0 - y\Big\rangle + \fint_0^{h_n} \langle x_t', x_t - x_0\rangle\,\mathrm{d}t \overset{n\to\infty}{\longrightarrow} \langle v, x_0 - y\rangle.$$

Therefore we finally conclude that

$$E(x_0) + \langle v, x_0 - y\rangle \leq \varliminf_{n\to\infty} \fint_0^{h_n} E(x_t) + \langle x_t', x_t - y\rangle\,\mathrm{d}t \leq E(y),$$

which proves that $v \in \partial^- E(x_0)$, as required. $\square$

Definition 25.1 (Heat flow) *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be an infinitesimally Hilbertian metric measure space. Then for any* $f \in L^2(\mathfrak{m})$ *and* $t \geq 0$*, we denote by* $\mathsf{h}_t f$ *the gradient flow of the Cheeger energy* $g \mapsto \frac{1}{2}\int |\mathrm{d}g|^2\,\mathrm{d}\mathfrak{m}$ *on* $L^2(\mathfrak{m})$ *(starting from* f*, at time* t*) We shall call it* heat flow*. This defines a family* $(\mathsf{h}_t)_{t\geq 0}$ *of operators* $\mathsf{h}_t : L^2(\mathfrak{m}) \to L^2(\mathfrak{m})$*.*

Proposition 25.2 *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be infinitesimally Hilbertian. Then the following hold:*

i) *The operator* $\mathsf{h}_t : L^2(\mathfrak{m}) \to L^2(\mathfrak{m})$ *is linear for every* $t \geq 0$*.*

ii) *For every* $f \in L^2(\mathfrak{m})$ *and* $t > 0$*, it holds that* $\mathsf{h}_t f \in D(\Delta)$ *and*

$$\frac{\mathsf{h}_{t+\varepsilon} f - \mathsf{h}_t f}{\varepsilon} \to \Delta \mathsf{h}_t f \quad \text{in } L^2(\mathfrak{m}) \quad \text{as } \varepsilon \to 0. \tag{25.5}$$

The same holds also at $t=0$ *provided* $f \in D(\Delta)$*.*

Proof. i) It directly follows from Theorem 24.8 and the fact that Δ is a linear operator.
ii) We know from Proposition 23.7 and Theorem 24.8 that $\mathsf{h}_t f \in D(\partial^- E) = D(\Delta)$ for every $t > 0$, thus it is sufficient to prove the claim for the case $t=0$ and $f \in D(\Delta)$. In this case we have $\partial^- E(f) = \{-\Delta f\}$ and thus the conclusion follows from by 7) of Theorem 24.8. $\square$

Proposition 25.3 (Δ and h_t commute) *Let* $f \in D(\Delta)$*. Then* $\mathsf{h}_t \Delta f = \Delta \mathsf{h}_t f$ *for all* $t \geq 0$*.*

Proof. Notice that

$$\Delta \mathsf{h}_t f = \lim_{\varepsilon\searrow 0} \frac{\mathsf{h}_t(\mathsf{h}_\varepsilon f) - \mathsf{h}_t f}{\varepsilon} = \mathsf{h}_t \left(\lim_{\varepsilon\searrow 0} \frac{\mathsf{h}_\varepsilon f - f}{\varepsilon} \right) = \mathsf{h}_t \Delta f,$$

which proves the statement. $\square$

Proposition 25.4 (Δ is self-adjoint) *Let* $f, g \in D(\Delta)$*. Then*

$$\int g\,\Delta f\,\mathrm{d}\mathfrak{m} = \int f\,\Delta g\,\mathrm{d}\mathfrak{m}. \tag{25.6}$$

Proof. Just notice that $\int g\,\Delta f\,\mathrm{d}\mathfrak{m} = \int \nabla f \cdot \nabla g\,\mathrm{d}\mathfrak{m} = \int f\,\Delta g\,\mathrm{d}\mathfrak{m}$. $\square$

Corollary 25.5 (h_t is self-adjoint) *Let $f, g \in L^2(\mathfrak{m})$ and $t \geq 0$. Then*

$$\int g\, \mathsf{h}_t f \,\mathrm{d}\mathfrak{m} = \int f\, \mathsf{h}_t g \,\mathrm{d}\mathfrak{m}. \tag{25.7}$$

Proof. Define $F(s) := \int \mathsf{h}_s f\, \mathsf{h}_{t-s} g \,\mathrm{d}\mathfrak{m}$ for every $s \in [0,t]$. Then the function F is AC and

$$F'(s) = \int \Delta \mathsf{h}_s f\, \mathsf{h}_{t-s} g - \mathsf{h}_s f\, \Delta \mathsf{h}_{t-s} g \,\mathrm{d}\mathfrak{m} \overset{(25.6)}{=} 0 \qquad \text{for a.e. } s \in [0,t],$$

whence accordingly $\int g\, \mathsf{h}_t f \,\mathrm{d}\mathfrak{m} = F(t) = F(0) = \int f\, \mathsf{h}_t g \,\mathrm{d}\mathfrak{m}$. □

Proposition 25.6 *Let $f \in L^2(\mathfrak{m})$. Then we have $f \in D(\Delta)$ if and only if $\frac{\mathsf{h}_t f - f}{t}$ admits a strong limit $g \in L^2(\mathfrak{m})$ as $t \searrow 0$. In this case, it holds that $g = \Delta f$.*

Proof. Necessity. Already established in point (ii) of Proposition 25.2.
Sufficiency. Suppose that $\frac{\mathsf{h}_t f - f}{t} \to g$ in $L^2(\mathfrak{m})$ as $t \searrow 0$. We first claim that $f \in W^{1,2}(\mathrm{X})$ and to this aim notice that for every $\varepsilon > 0$, our assumption and the self-adjointness of h_ε give

$$\int \mathsf{h}_\varepsilon f g \,\mathrm{d}\mathfrak{m} = \lim_{t \downarrow 0} \int \mathsf{h}_\varepsilon f \frac{\mathsf{h}_t f - f}{t} \,\mathrm{d}\mathfrak{m} = \lim_{t \downarrow 0} \int f \frac{\mathsf{h}_t \mathsf{h}_\varepsilon f - \mathsf{h}_\varepsilon f}{t} \,\mathrm{d}\mathfrak{m}$$

and since $\mathsf{h}_\varepsilon f \in D(\Delta)$, the 'necessity' proved before and the fact that the heat flow commutes with Δ give

$$\int \mathsf{h}_\varepsilon f\, g \,\mathrm{d}\mathfrak{m} = \int f\, \Delta \mathsf{h}_\varepsilon f \,\mathrm{d}\mathfrak{m} = \int \mathsf{h}_{\varepsilon/2} f\, \Delta \mathsf{h}_{\varepsilon/2} f \,\mathrm{d}\mathfrak{m} = -\int |\nabla \mathsf{h}_{\varepsilon/2} f|^2 \,\mathrm{d}\mathfrak{m}.$$

Since $f \in L^2(\mathrm{X})$, the (absolute value of the) leftmost side of this last identity remains bounded as $\varepsilon \downarrow 0$, hence the same holds for the rightmost one. Thus the lower semicontinuity of the Cheeger energy E gives

$$E(f) \leq \varliminf_{\varepsilon \downarrow 0} E(\mathsf{h}_\varepsilon f) = \varliminf_{\varepsilon \downarrow 0} \frac{1}{2} \int |\nabla \mathsf{h}_\varepsilon f|^2 \,\mathrm{d}\mathfrak{m} < \infty,$$

thus giving our claim $f \in W^{1,2}(\mathrm{X})$. Now observe that the inequality $E(\mathsf{h}_s f) \leq E(f)$, valid for all $s \geq 0$, ensures that $(\mathsf{h}_\varepsilon f)$ is bounded in $W^{1,2}(\mathrm{X})$ and thus weakly relatively compact. Since $\mathsf{h}_\varepsilon f \to f$ in $L^2(\mathrm{X})$ as $\varepsilon \searrow 0$, we deduce that $\mathsf{h}_\varepsilon f \rightharpoonup f$ weakly in $W^{1,2}(\mathrm{X})$. Given any $\ell \in W^{1,2}(\mathrm{X})$, we thus have that

$$\begin{aligned}\int g\, \ell \,\mathrm{d}\mathfrak{m} &= \lim_{t \searrow 0} \int \frac{\mathsf{h}_t f - f}{t}\, \ell \,\mathrm{d}\mathfrak{m} = \lim_{t \searrow 0} \fint_0^t \int \Delta \mathsf{h}_s f\, \ell \,\mathrm{d}\mathfrak{m} \,\mathrm{d}s = -\lim_{t \searrow 0} \fint_0^t \int \nabla \mathsf{h}_s f \cdot \nabla \ell \,\mathrm{d}\mathfrak{m} \,\mathrm{d}s \\ &= -\int \nabla f \cdot \nabla \ell \,\mathrm{d}\mathfrak{m},\end{aligned}$$

which shows that $f \in D(\Delta)$ and $\Delta f = g$. □

26 Lesson [19/02/2018]

Remark 26.1 There exists a universal constant $C > 0$ such that

$$E(\mathsf{h}_t f) \le C\,\frac{\|f\|^2_{L^2(\mathfrak{m})}}{t} \quad \text{and} \quad \|\Delta \mathsf{h}_t f\|^2_{L^2(\mathfrak{m})} \le C\,\frac{\|f\|^2_{L^2(\mathfrak{m})}}{t^2} \qquad \text{for all } f \in L^2(\mathfrak{m}) \text{ and } t > 0. \tag{26.1}$$

Such claim directly follows from item 6) of Theorem 24.8. ■

Proposition 26.2 *Let $f \in L^2(\mathfrak{m})$ be fixed. Then the following hold:*

i) *The map $(0,+\infty) \ni t \mapsto \mathsf{h}_t f$ belongs to $C^\infty\big((0,+\infty), W^{1,2}(\mathrm{X})\big)$.*

ii) *It holds that $\mathsf{h}_t f \in D(\Delta^{(n)})$ for every $n \in \mathbb{N}$ and $t > 0$.*

Proof. i) Fix $\varepsilon > 0$. It suffices to prove that $t \mapsto \mathsf{h}_t f$ belongs to $C^1\big((\varepsilon,+\infty), W^{1,2}(\mathrm{X})\big)$. Recall that $\frac{\mathrm{d}}{\mathrm{d}t}\,\mathsf{h}_t f = \Delta \mathsf{h}_t f$ for a.e. $t > \varepsilon$ and that the map $(\varepsilon,+\infty) \ni t \mapsto \Delta \mathsf{h}_t f = \mathsf{h}_{t-\varepsilon}\Delta \mathsf{h}_\varepsilon f \in L^2(\mathfrak{m})$ is continuous. Call $g := \Delta \mathsf{h}_\varepsilon f$. Since even the map

$$(\varepsilon,+\infty) \ni t \longmapsto \int |\nabla \mathsf{h}_{t-\varepsilon} g|^2 \,\mathrm{d}\mathfrak{m} = -\int \mathsf{h}_{t-\varepsilon} g\, \Delta \mathsf{h}_{t-\varepsilon} g \,\mathrm{d}\mathfrak{m}$$

is continuous, we conclude that $(\varepsilon,+\infty) \ni t \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\,\mathsf{h}_t f = \mathsf{h}_{t-\varepsilon} g \in W^{1,2}(\mathrm{X})$ is continuous as well. This grants that $\big(t \mapsto \mathsf{h}_t f\big) \in C^1\big((\varepsilon,+\infty), W^{1,2}(\mathrm{X})\big)$.
ii) It suffices to show that $\Delta \mathsf{h}_t f \in D(\Delta)$ for all $f \in L^2(\mathfrak{m})$ and $t > 0$. This immediately follows from the fact that $\Delta \mathsf{h}_t f = \mathsf{h}_{t/2}\Delta \mathsf{h}_{t/2} f \in D(\Delta)$. □

Lemma 26.3 *Let $u : \mathbb{R} \to [0,+\infty]$ be a convex lsc function such that $u(0) = 0$. Define*

$$\mathcal{C} := \Big\{ v \in C^\infty(\mathbb{R}) \;\Big|\; v \ge 0 \text{ is convex, } v(0) = v'(0) = 0,\ v', v'' \text{ are bounded} \Big\}.$$

Then there exists a sequence $(u_n)_n \subseteq \mathcal{C}$ such that $u_n(t) \nearrow u(t)$ for all $t \in \mathbb{R}$.

Proof. Let us define $\tilde{u}(t) := \sup\big\{ v(t) \,\big|\, v \in \mathcal{C},\, v \le u \big\} \le u(t)$ for all $t \in \mathbb{R}$. It can be readily checked that actually $\tilde{u} = u$. Now call $I := \{u < +\infty\}$ and fix any compact interval $K \subseteq I$ such that $\mathrm{dist}(K, \mathbb{R} \setminus I) > 0$. Then there exists a constant $C(K,u) > 0$ such that each $v \in \mathcal{C}$ with $v \le u$ is $C(K,u)$-Lipschitz in K. Moreover, for a suitable sequence $(v_n)_n \subseteq \mathcal{C}$ we have that $\operatorname{ess\,sup}\big\{ v \in \mathcal{C} \,:\, v \le u \big\} = \sup_n v_n$ holds a.e. in K. These two facts grant that actually the equality $\tilde{u} = \sup_n v_n$ holds everywhere in K. Since $\mathrm{int}(I)$ can be written as countable union of intervals K as above, we deduce that there exists $(w_n)_n \subseteq \mathcal{C}$ such that $\tilde{u} = \sup_n w_n$. Finally, we would like to define $u_n := \max_{i \le n} w_i$ for all $n \in \mathbb{N}$, but such functions have all the required properties apart from smoothness. Therefore the desired functions u_n can be easily built by recalling the facts that $\max\{w_1, w_2\} = \frac{1}{2}\big(|w_1 - w_2| + w_1 + w_2\big)$ and that for all $t \in \mathbb{R}$ one has $|w_1 - w_2|(t) = \sup_{\varepsilon > 0} \sqrt{|w_1 - w_2|^2(t) + \varepsilon^2} - \varepsilon$. □

Proposition 26.4 *Let $f \in L^2(\mathfrak{m})$ be fixed. Then the following hold:*

i) Weak maximum principle. *Suppose that $f \le c$ holds $\mathfrak{m}$-a.e. for some constant $c \in \mathbb{R}$. Then $\mathsf{h}_t f \le c$ holds $\mathfrak{m}$-a.e. for every $t > 0$.*

ii) *Let $u : \mathbb{R} \to [0, +\infty]$ be any convex lower semicontinuous function satisfying $u(0) = 0$. Then the function $[0, +\infty) \ni t \mapsto \int u(\mathsf{h}_t f)\,\mathrm{d}\mathfrak{m}$ is non-increasing.*

iii) *Let $p \in [1, \infty]$ be given. Then $\|\mathsf{h}_t f\|_{L^p(\mathfrak{m})} \le \|f\|_{L^p(\mathfrak{m})}$ holds for every $t > 0$.*

Proof. i) By recalling the 'minimising movements' technique that we used in Step 1 of Theorem 24.8 to prove existence of the gradient flow, one can easily realise that it is enough to show that for any $\tau > 0$ the minimum f_τ of $g \mapsto E(g) + \|f - g\|^2_{L^2(\mathfrak{m})}/(2\,\tau)$ is $\mathfrak{m}$-a.e. smaller than of equal to c. We argue by contradiction: if not, then $\bar{f} := f_\tau \wedge c$ would satisfy the inequalities $E(\bar{f}) \le E(f_\tau)$ and $\|f - \bar{f}\|_{L^2(\mathfrak{m})} < \|f - f_\tau\|_{L^2(\mathfrak{m})}$, thus contradicting the minimality of f_τ. Hence the weak maximum principle i) is proved.
ii) First of all, we prove it for $u \in C^\infty(\mathbb{R})$ such that $u(0) = u'(0) = 0$ and u', u'' are bounded. Say $|u'(t)|, |u''(t)| \le C$ for all $t \in \mathbb{R}$. For any $t \ge s$, we thus have that

$$
\begin{aligned}
\big|u(t) - u(s)\big| &= \left|\int_s^t u'(r)\,\mathrm{d}r\right| = \left|(t-s)\,u'(s) + \int_s^t \big(u'(r) - u'(s)\big)\,\mathrm{d}r\right| \\
&\le C\,|s|\,(t-s) + \int_s^t\!\!\int_s^r u''(r')\,\mathrm{d}r'\,\mathrm{d}r \le C\,\big[(t-s)^2 + |s|\,(t-s)\big].
\end{aligned}
\tag{26.2}
$$

Given that $(0, +\infty) \ni t \mapsto \mathsf{h}_t f \in L^2(\mathfrak{m})$ is locally Lipschitz, we deduce from (26.2) that the function $t \mapsto \int u(\mathsf{h}_t f)\,\mathrm{d}\mathfrak{m}$, which is continuous on $[0, +\infty)$, is locally Lipschitz on $(0, +\infty)$. By passing to the limit as $\varepsilon \searrow 0$ in the equalities

$$
\int \frac{u(\mathsf{h}_{t+\varepsilon} f) - u(\mathsf{h}_t f)}{\varepsilon}\,\mathrm{d}\mathfrak{m} = \int\!\!⨍_t^{t+\varepsilon} u'(\mathsf{h}_s f)\,\Delta \mathsf{h}_s f\,\mathrm{d}\mathfrak{m} = \int_0^1\!\!\int u'(\mathsf{h}_{t+\varepsilon r} f)\,\Delta \mathsf{h}_{t+\varepsilon r} f\,\mathrm{d}\mathfrak{m}\,\mathrm{d}r,
$$

we see that $\frac{\mathrm{d}}{\mathrm{d}t}\int u(\mathsf{h}_t f)\,\mathrm{d}\mathfrak{m} = \int u'(\mathsf{h}_t f)\,\Delta \mathsf{h}_t f\,\mathrm{d}\mathfrak{m}$ for a.e. $t > 0$. Hence by using the chain rule for the differential and the fact that $u'' \ge 0$ we finally conclude that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int u(\mathsf{h}_t f)\,\mathrm{d}\mathfrak{m} &= \int u'(\mathsf{h}_t f)\,\Delta \mathsf{h}_t f\,\mathrm{d}\mathfrak{m} = -\int \nabla u'(\mathsf{h}_t f)\cdot \nabla \mathsf{h}_t f\,\mathrm{d}\mathfrak{m} \\
&= -\int u''(\mathsf{h}_t f)\,|\nabla \mathsf{h}_t f|^2\,\mathrm{d}\mathfrak{m} \le 0 \qquad \text{for a.e. } t > 0,
\end{aligned}
$$

which ensures that the function $[0, +\infty) \ni t \mapsto \int u(\mathsf{h}_t f)\,\mathrm{d}\mathfrak{m}$ is non-increasing.

Now consider the case of a general function u. Consider an approximating sequence $(u_n)_n$ as in Lemma 26.3. By monotone convergence theorem, we thus see that

$$
\int u(\mathsf{h}_t f)\,\mathrm{d}\mathfrak{m} = \sup_{n \in \mathbb{N}} \int u_n(\mathsf{h}_t f)\,\mathrm{d}\mathfrak{m} \qquad \text{for every } t \ge 0.
$$

Hence $t \mapsto \int u(\mathsf{h}_t f)\,\mathrm{d}\mathfrak{m}$ is non-increasing as pointwise supremum of non-increasing functions.
iii) To prove the statement for $p \in [1, \infty)$, just apply ii) with $u := |\cdot|^p$. For the case $p = \infty$, notice that $-\|f\|_{L^\infty(\mathfrak{m})} \le f \le \|f\|_{L^\infty(\mathfrak{m})}$ holds $\mathfrak{m}$-a.e., whence $-\|f\|_{L^\infty(\mathfrak{m})} \le \mathsf{h}_t f \le \|f\|_{L^\infty(\mathfrak{m})}$ holds $\mathfrak{m}$-a.e. for every $t > 0$ by i), so that $\|\mathsf{h}_t f\|_{L^\infty(\mathfrak{m})} \le \|f\|_{L^\infty(\mathfrak{m})}$ for all $t > 0$. $\square$

Proposition 26.5 (Heat flow in $L^p(\mathfrak{m})$) *Let $p \in [1,\infty)$ be given. Then the heat flow uniquely extends to a family of linear contractions in $L^p(\mathfrak{m})$.*

Proof. It follows from Proposition 26.4 and the density of $L^2(\mathfrak{m}) \cap L^p(\mathfrak{m})$ in $L^p(\mathfrak{m})$. □

Definition 26.6 (Heat flow in $L^\infty(\mathfrak{m})$) *Let $f \in L^\infty(\mathfrak{m})$ be given. Then for every $t > 0$ we define $\mathsf{h}_t f \in L^\infty(\mathfrak{m})$ as the function corresponding to $\big[L^1(\mathfrak{m}) \ni g \mapsto \int f\, \mathsf{h}_t g \,\mathrm{d}\mathfrak{m} \in \mathbb{R}\big] \in L^1(\mathfrak{m})'$.*

Notice that the previous definition is well-posed because $\big|\int f\, \mathsf{h}_t g \,\mathrm{d}\mathfrak{m}\big| \leq \|f\|_{L^\infty(\mathfrak{m})} \|g\|_{L^1(\mathfrak{m})}$ is verified by item iii) of Proposition 26.4.

Exercise 26.7 Given any $p \in [1,\infty]$ and $t > 0$, we (provisionally) denote by h_t^p the heat flow in $L^p(\mathfrak{m})$ at time t. Prove that $\mathsf{h}_t^p f = \mathsf{h}_t^q f$ for every $p, q \in [1,\infty]$ and $f \in L^p(\mathfrak{m}) \cap L^q(\mathfrak{m})$.

Proposition 26.8 *Let $\varphi \in C_c^\infty(0,+\infty)$ and $p \in [1,\infty]$ be given. For any $f \in L^2(\mathfrak{m}) \cap L^p(\mathfrak{m})$, let us define $\mathsf{h}_\varphi f \in L^2(\mathfrak{m}) \cap L^p(\mathfrak{m})$ as*

$$\mathsf{h}_\varphi f := \int_0^{+\infty} \mathsf{h}_s f\, \varphi(s) \,\mathrm{d}s. \tag{26.3}$$

Then $\mathsf{h}_\varphi f \in D(\Delta)$ and $\|\Delta \mathsf{h}_\varphi f\|_{L^p(\mathfrak{m})} \leq C(\varphi)\, \|f\|_{L^p(\mathfrak{m})}$ for some constant $C(\varphi) > 0$.

Proof. By applying Theorem 6.8, we see that $\mathsf{h}_\varphi f \in D(\Delta)$ and that

$$\Delta \mathsf{h}_\varphi f = \int_0^{+\infty} \Delta \mathsf{h}_s f\, \varphi(s) \,\mathrm{d}s = \int_0^{+\infty} \frac{\mathrm{d}}{\mathrm{d}s} \mathsf{h}_s f\, \varphi(s) \,\mathrm{d}s = -\int_0^{+\infty} \mathsf{h}_s f\, \varphi'(s) \,\mathrm{d}s,$$

whence accordingly item iii) of Proposition 26.4 yields

$$\|\Delta \mathsf{h}_\varphi f\|_{L^p(\mathfrak{m})} \leq \int_0^{+\infty} \|\mathsf{h}_s f\|_{L^p(\mathfrak{m})} |\varphi'|(s) \,\mathrm{d}s \leq \|f\|_{L^p(\mathfrak{m})} \int_0^{+\infty} |\varphi'|(s) \,\mathrm{d}s.$$

Therefore the statement is verified with $C(\varphi) := \int_0^{+\infty} |\varphi'|(s) \,\mathrm{d}s$. □

A direct consequence of Proposition 26.8 is given by the next result:

Corollary 26.9 *The family $\big\{f \in L^2(\mathfrak{m}) \cap L^\infty(\mathfrak{m}) \;\big|\; f \geq 0,\, f \in D(\Delta),\, \Delta f \in W^{1,2}(\mathrm{X})\big\}$ is strongly $L^2(\mathfrak{m})$-dense in $\big\{f \in L^2(\mathfrak{m}) \;\big|\; f \geq 0\big\}$.*

27 Lesson [21/02/2018]

Consider any smooth function $f : \mathbb{R}^d \to \mathbb{R}$. An easy computation yields the following formula:

$$\Delta \frac{|\nabla f|^2}{2} = |Hf|_{\mathsf{HS}}^2 + \nabla f \cdot \nabla \Delta f. \tag{27.1}$$

Now consider any smooth Riemannian manifold (M, g). Recall that the *Riemann curvature tensor* is given by

$$\mathrm{R}(X,Y,Z,W) := \big\langle \nabla_X \nabla_Y Z - \nabla_Y \nabla_{\mathrm{X}} Z - \nabla_{[X,Y]} Z, W \big\rangle,$$

while the *Ricci curvature tensor* is defined as

$$\mathrm{Ric}(X,Y) := \sum_{i=1}^{\dim M} \mathrm{R}(e_i, X, Y, e_i)$$

where $(e_i)_i$ is any (local) *frame*, i.e. a family of vector fields that form an orthonormal basis of the tangent space at all points.

Observe that in 27.1 three derivatives of f appear, thus an analogous formula for M should contain a correction term due to the presence of the curvature. Indeed, it turns out that for any $f \in C^\infty(M)$ we have

$$\Delta \frac{|\nabla f|^2}{2} = |Hf|^2_{\mathsf{HS}} + \nabla f \cdot \nabla \Delta f + \mathrm{Ric}(\nabla f, \nabla f). \tag{27.2}$$

Formula (27.2) is called *Bochner identity.* In order to generalise the notion of 'having Ricci curvature greater than or equal to K' to the framework of metric measure spaces, we need the following simple result:

Proposition 27.1 *Let (M,g) be a smooth Riemannian manifold and let $K \in \mathbb{R}$. Then the following are equivalent:*

i) $\mathrm{Ric}_M \geq Kg$, *i.e. for any $p \in M$ and $v \in T_pM$ we have that* $\mathrm{Ric}_p(v,v) \geq K|v|^2$.

ii) *For any $f \in C^\infty(M)$ it holds that*

$$\Delta \frac{|\nabla f|^2}{2} \geq \nabla f \cdot \nabla \Delta f + K|\nabla f|^2, \tag{27.3}$$

which is called Bochner inequality.

Proof. The implication i) $\Longrightarrow$ ii) is trivial, then it just suffices to prove ii) $\Longrightarrow$ i). Suppose to have $p \in M$ and $v \in T_pM$ such that $\mathrm{Ric}_p(v,v) < K|v|^2$. Hence there exists $f \in C^\infty(M)$ satisfying $\nabla f_p = v$ and $Hf_p = 0$. Then $\Delta \frac{|\nabla f|^2}{2}(p) < \nabla f_p \cdot \nabla \Delta f_p + K|\nabla f_p|^2$, which is in contradiction with (27.3). $\square$

We are now in a position to give the definition of the $\mathsf{RCD}(K,\infty)$ condition:

Definition 27.2 (Ambrosio-Gigli-Savaré '11) *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space and let $K \in \mathbb{R}$. Then we say that $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is an* $\mathsf{RCD}(K,\infty)$ space *provided:*

i) *There exist $C > 0$ and $\bar{x} \in \mathrm{X}$ such that $\mathfrak{m}\big(B_r(\bar{x})\big) \leq \exp(Cr^2)$ for all $r > 0$.*

ii) *If $f \in W^{1,2}(\mathrm{X})$ satisfies $|Df| \in L^\infty(\mathfrak{m})$, then there exists $\tilde{f} \in \mathrm{LIP}(\mathrm{X})$ such that $\tilde{f} = f$ holds $\mathfrak{m}$-a.e. and $\mathrm{Lip}(\tilde{f}) = \big\||Df|\big\|_{L^\infty(\mathfrak{m})}$.*

iii) *$(\mathrm{X},\mathsf{d},\mathfrak{m})$ is infinitesimally Hilbertian.*

iv) *The* weak Bochner inequality *is satisfied, i.e.*

$$\int \Delta g \, \frac{|\nabla f|^2}{2} \, \mathrm{d}\mathfrak{m} \geq \int g \left[\nabla f \cdot \nabla \Delta f + K |\nabla f|^2 \right] \mathrm{d}\mathfrak{m} \tag{27.4}$$

for every choice of functions $f \in D(\Delta)$ *and* $g \in D(\Delta) \cap L^\infty(\mathfrak{m})^+$ *with* $\Delta f \in W^{1,2}(\mathrm{X})$ *and* $\Delta g \in L^\infty(\mathfrak{m})$.

Remark 27.3 Item ii) in Definition 27.2 is verified if and only if both these conditions hold:

a) If $f \in W^{1,2}(\mathrm{X})$ satisfies $|Df| \in L^\infty(\mathfrak{m})$, then there exists $\tilde{f} : \mathrm{X} \to \mathbb{R}$ locally Lipschitz such that $\tilde{f} = f$ holds $\mathfrak{m}$-a.e. in X and $\mathrm{lip}(\tilde{f}) \leq \big\| |Df| \big\|_{L^\infty(\mathfrak{m})}$.

b) If $\tilde{f} : \mathrm{X} \to \mathbb{R}$ is locally Lipschitz and $\mathrm{lip}(\tilde{f}) \leq L$, then $\tilde{f}$ is L-Lipschitz.

The role of ii) is to link the metric structure of the space with the reference measure. ■

Theorem 27.4 (Bakry-Émery estimate) *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K, \infty)$ *space, for some constant* $K \in \mathbb{R}$. *Consider* $f \in W^{1,2}(\mathrm{X})$ *and* $t \geq 0$. *Then*

$$|D\mathsf{h}_t f|^2 \leq e^{-2Kt} \, \mathsf{h}_t\big(|Df|^2\big) \qquad \textit{holds } \mathfrak{m}\textit{-a.e. in } \mathrm{X}. \tag{27.5}$$

Proof. Fix $g \in D(\Delta) \cap L^\infty(\mathfrak{m})^+$ such that $\Delta g \in L^\infty(\mathfrak{m})$ and $t > 0$. Define $F : [0, t] \to \mathbb{R}$ as

$$F(s) := \int \mathsf{h}_s g \, |D\mathsf{h}_{t-s} f|^2 \, \mathrm{d}\mathfrak{m} \qquad \text{for every } s \in [0, t].$$

Since $t \mapsto \mathsf{h}_t f \in W^{1,2}(\mathrm{X})$ is of class C^1 by Proposition 26.2, we know that $t \mapsto |D\mathsf{h}_t f|^2 \in L^1(\mathfrak{m})$ is of class C^1 as well. Moreover, from the $\mathfrak{m}$-a.e. inequality

$$|\mathsf{h}_t g - \mathsf{h}_s g| = \left| \int_s^t \frac{\mathrm{d}}{\mathrm{d}r} \, \mathsf{h}_r g \, \mathrm{d}r \right| \leq \int_s^t |\Delta \mathsf{h}_r g| \, \mathrm{d}r = \int_s^t |\mathsf{h}_r \Delta g| \, \mathrm{d}r \leq |t - s| \, \|\Delta g\|_{L^\infty(\mathfrak{m})},$$

which is granted by Proposition 25.3 and the weak maximum principle, we immediately deduce that $\|\mathsf{h}_t g - \mathsf{h}_s g\|_{L^\infty(\mathfrak{m})} \leq |t - s| \, \|\Delta g\|_{L^\infty(\mathfrak{m})}$, in other words $t \mapsto \mathsf{h}_t g \in L^\infty(\mathfrak{m})$ is Lipschitz. Therefore F is Lipschitz and it holds that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s} F(s) &= \int \Delta \mathsf{h}_s g \, |D\mathsf{h}_{t-s} f|^2 - 2 \, \mathsf{h}_s g \, \nabla \mathsf{h}_{t-s} f \cdot \nabla \Delta \mathsf{h}_{t-s} f \, \mathrm{d}\mathfrak{m} \overset{(27.4)}{\geq} 2 \, K \int \mathsf{h}_s g \, |D\mathsf{h}_{t-s} f|^2 \, \mathrm{d}\mathfrak{m} \\ &= 2 \, K \, F(s) \qquad \text{for a.e. } s \in [0, t]. \end{aligned}$$

Hence Gronwall lemma grants that $F(t) \geq e^{2Kt} F(0)$, or equivalently

$$\int g \, |D\mathsf{h}_t f|^2 \, \mathrm{d}\mathfrak{m} \leq e^{-2Kt} \int g \, \mathsf{h}_t\big(|Df|^2\big) \, \mathrm{d}\mathfrak{m}.$$

Since the class of g's under consideration is weakly*-dense in $\big\{ g \in L^\infty(\mathfrak{m}) : g \geq 0 \big\}$, we finally conclude that (27.5) is satisfied. □

28 Lesson [21/02/2018]

From now on, $(\mathrm{X},\mathsf{d},\mathfrak{m})$ will always be an $\mathsf{RCD}(K,\infty)$ space, for some $K\in\mathbb{R}$.

Lemma 28.1 *Let $f,g\in D(\Delta)\cap L^\infty(\mathfrak{m})$ be given. Then*

$$\int \Delta g\,\frac{f^2}{2}\,\mathrm{d}\mathfrak{m}=\int g\big(f\,\Delta f+|Df|^2\big)\,\mathrm{d}\mathfrak{m}. \tag{28.1}$$

Proof. Since $fg\in W^{1,2}(\mathrm{X})$, we see that

$$\begin{aligned}\int fg\,\Delta f\,\mathrm{d}\mathfrak{m}&=-\int\nabla(fg)\cdot\nabla f\,\mathrm{d}\mathfrak{m}=-\int|Df|^2+f\,\nabla g\cdot\nabla f\,\mathrm{d}\mathfrak{m}\\&=-\int g\,|Df|^2+\nabla\frac{f^2}{2}\cdot\nabla g\,\mathrm{d}\mathfrak{m},\end{aligned}$$

which gives the statement. □

Proposition 28.2 (L^∞-Lip regularisation of the heat flow) *Let $f\in L^\infty(\mathfrak{m})$ and $t>0$ be given. Then $|D\mathsf{h}_tf|\in L^\infty(\mathfrak{m})$ and*

$$\big\||D\mathsf{h}_tf|\big\|_{L^\infty(\mathfrak{m})}\leq\frac{C(k)}{\sqrt{t}}\,\|f\|_{L^\infty(\mathfrak{m})}\qquad\textit{provided } t\in(0,1). \tag{28.2}$$

In particular, the function h_tf admits a Lipschitz representative.

Proof. It suffices to prove the statement for $f\in L^2(\mathfrak{m})\cap L^\infty(\mathfrak{m})$. Fix any $g\in D(\Delta)\cap L^\infty(\mathfrak{m})^+$ such that $\Delta g\in L^\infty(\mathfrak{m})$. Take $t\in(0,1)$ and define $F:\,[0,t]\to\mathbb{R}$ as

$$F(s):=\int\mathsf{h}_sg\,|\mathsf{h}_{t-s}f|^2\,\mathrm{d}\mathfrak{m}\qquad\text{for every } s\in[0,t].$$

We already know that $F\in C^0\big([0,t]\big)\cap C^1\big((0,t)\big)$ and that for a.e. $s\in[0,t]$ it holds

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}F(s)&=\int\Delta\mathsf{h}_sg\,|\mathsf{h}_{t-s}f|^2-2\,\mathsf{h}_sg\,\mathsf{h}_{t-s}f\,\Delta\mathsf{h}_{t-s}f\,\mathrm{d}\mathfrak{m}\overset{(28.1)}{=}2\int\mathsf{h}_sg\,|D\mathsf{h}_{t-s}f|^2\,\mathrm{d}\mathfrak{m}\\&=2\int g\,\mathsf{h}_s|D\mathsf{h}_{t-s}f|^2\,\mathrm{d}\mathfrak{m}\overset{(27.5)}{\geq}2\,C(k)\int g\,|D\mathsf{h}_tf|^2\mathrm{d}\mathfrak{m}.\end{aligned}$$

By integrating the previous inequality on $[0,t]$, we obtain that

$$2\,C(k)\,t\int g\,|D\mathsf{h}_tf|^2\,\mathrm{d}\mathfrak{m}\leq F(t)-F(0)\leq\int g\,\mathsf{h}_t(f^2)\,\mathrm{d}\mathfrak{m}.$$

By the weak*-density of such g's, we see that the inequality $2\,C(k)\,t\,|D\mathsf{h}_tf|^2\leq\mathsf{h}_t(f^2)$ holds $\mathfrak{m}$-a.e. in X. Therefore the weak maximum principle grants that (28.2) is satisfied. Finally, the last statement immediately follows from item ii) of Definition 27.2. □

Definition 28.3 (Savaré '14) *Let us define*

$$\begin{aligned}\mathrm{Test}^\infty(\mathrm{X}) &:= \Big\{f \in \mathrm{LIP}(\mathrm{X}) \cap L^\infty(\mathfrak{m}) \cap D(\Delta) \,\Big|\, \Delta f \in W^{1,2}(\mathrm{X}) \cap L^\infty(\mathfrak{m})\Big\},\\ \mathrm{Test}^\infty_+(\mathrm{X}) &:= \big\{f \in \mathrm{Test}^\infty(\mathrm{X}) \,\big|\, f \geq 0 \text{ holds } \mathfrak{m}\text{-a.e. on } \mathrm{X}\big\}.\end{aligned} \tag{28.3}$$

Proposition 28.4 *The space* $\mathrm{Test}^\infty_+(\mathrm{X})$ *is dense in* $W^{1,2}(\mathrm{X})^+$*. Moreover, the space* $\mathrm{Test}^\infty(\mathrm{X})$ *is dense in* $W^{1,2}(\mathrm{X})$*.*

Proof. Let $f \in W^{1,2}(\mathrm{X})^+$ be fixed. Call $f_n := f \wedge n \in W^{1,2}(\mathrm{X})^+ \cap L^\infty(\mathfrak{m})$ for any $n \in \mathbb{N}$, so that $f_n \to f$ in $W^{1,2}(\mathrm{X})$. Then it suffices to prove that each f_n belongs to the $W^{1,2}(\mathrm{X})$-closure of $\mathrm{Test}^\infty_+(\mathrm{X})$. We now claim that

$$\mathsf{h}_\varphi f_n \in \mathrm{Test}^\infty_+(\mathrm{X}) \qquad \text{for every } \varphi \in C^\infty_c(0,+\infty). \tag{28.4}$$

We have that $\mathsf{h}_\varphi f_n \geq 0$ holds $\mathfrak{m}$-a.e. by the weak maximum principle. By arguing as in Proposition 26.8, we also see that $\mathsf{h}_\varphi f_n \in D(\Delta)\cap L^\infty(\mathfrak{m})$. Choose $\varepsilon \in (0,1)$ so that the support of φ is contained in $[\varepsilon,\varepsilon^{-1}]$, then the fact that $\Delta \mathsf{h}_t f_n = \mathsf{h}_{t-\varepsilon/2}\Delta \mathsf{h}_{\varepsilon/2} f_n$ for all $t \geq \varepsilon$ can be used to prove that $\Delta \mathsf{h}_\varphi f_n \in W^{1,2}(\mathrm{X})\cap L^\infty(\mathfrak{m})$. Finally, it holds that $\mathsf{h}_\varphi f_n \in \mathrm{LIP}(\mathrm{X})$ by Proposition 28.2. Therefore the claim 28.4 is proved. Now take any sequence $(\varphi_k)_k \subseteq C^\infty_c(0,+\infty)$ such that $\varphi_k \rightharpoonup \delta_0$. Then $\mathsf{h}_{\varphi_k} f_n \to f_n$ strongly in $W^{1,2}(\mathrm{X})$, proving that each f_n is in the closure of the space $\mathrm{Test}^\infty_+(\mathrm{X})$, as required.

The second statement follows from the first one by noticing that for every $f \in W^{1,2}(\mathrm{X})$ it holds that $f = f^+ - f^-$ and $f^\pm \in W^{1,2}(\mathrm{X})^+$. □

By making use of the assumed lower Ricci curvature bounds, we can prove the following regularity of the minimal weak upper gradients of the test functions:

Lemma 28.5 *Let* $f \in \mathrm{Test}^\infty(\mathrm{X})$ *be given. Then* $|Df|^2 \in W^{1,2}(\mathrm{X})$*.*

Proof. Given any $g \in D(\Delta) \cap L^\infty(\mathfrak{m})^+$ and any sequence $(\varphi_k)_k \subseteq C^\infty_c(0,+\infty)$ with $\varphi_k \rightharpoonup \delta_0$, we deduce from Proposition 26.8 that $\mathsf{h}_{\varphi_k} g \rightharpoonup g$ weakly* in $L^\infty(\mathfrak{m})$ and $L^\infty(\mathfrak{m}) \ni \Delta \mathsf{h}_{\varphi_k} g \to \Delta g$ in $L^2(\mathfrak{m})$. Thus taking into account item iv) of Definition 27.2 and the fact that $|\nabla f|^2 \in L^2(\mathfrak{m})$, we see that

$$\frac{1}{2}\int \Delta g\,|\nabla f|^2 \,\mathrm{d}\mathfrak{m} \geq \int g\,\big(\nabla f \cdot \nabla \Delta f + K|\nabla f|^2\big)\,\mathrm{d}\mathfrak{m} \qquad \text{for every } g \in D(\Delta)\cap L^\infty(\mathfrak{m})^+. \tag{28.5}$$

Now we apply (28.5) with $g := \mathsf{h}_t\big(|\nabla f|^2\big)$, obtaining that:

$$\begin{aligned} E\big(|\nabla f|^2\big) &\leq \varliminf_{t\searrow 0} E\big(\mathsf{h}_{t/2}|\nabla f|^2\big) = \varliminf_{t\searrow 0}\frac{1}{2}\int \big|\nabla \mathsf{h}_{t/2}|\nabla f|^2\big|^2\,\mathrm{d}\mathfrak{m}\\ &= -\varliminf_{t\searrow 0}\frac{1}{2}\int \mathsf{h}_{t/2}\big(|\nabla f|^2\big)\,\Delta \mathsf{h}_{t/2}\big(|\nabla f|^2\big)\,\mathrm{d}\mathfrak{m}\\ &= -\varliminf_{t\searrow 0}\frac{1}{2}\int \Delta \mathsf{h}_t\big(|\nabla f|^2\big)\,|\nabla f|^2\,\mathrm{d}\mathfrak{m}\\ &\leq -\varliminf_{t\searrow 0}\int \mathsf{h}_t\big(|\nabla f|^2\big)\,\big(\nabla f\cdot\nabla\Delta f + K|\nabla f|^2\big)\,\mathrm{d}\mathfrak{m}\\ &\leq \mathrm{Lip}(f)^2\int \big(\nabla f\cdot\nabla\Delta f + K|\nabla f|^2\big)\,\mathrm{d}\mathfrak{m} < +\infty,\end{aligned}$$

whence $|Df|^2 \in W^{1,2}(\mathrm{X})$, as required. $\square$

Remark 28.6 Given any $f \in \mathrm{Test}^\infty(\mathrm{X})$, it holds that

$$E\big(|Df|^2\big) \leq \mathrm{Lip}(f)^2\, \|f\|_{W^{1,2}(\mathrm{X})}\, \big(\|\Delta f\|_{W^{1,2}(\mathrm{X})} + \|f\|_{W^{1,2}(\mathrm{X})}\big), \tag{28.6}$$

as a consequence of the estimates in the proof of Lemma 28.5. $\blacksquare$

Theorem 28.7 (Savaré '14) *The space* $\mathrm{Test}^\infty(\mathrm{X})$ *is an algebra.*

Proof. Fix $f, g \in \mathrm{Test}^\infty(\mathrm{X})$. We aim to prove that $fg \in \mathrm{Test}^\infty(\mathrm{X})$ as well. It is immediate to check that $fg \in \mathrm{LIP}(\mathrm{X}) \cap L^\infty(\mathfrak{m})$. Moreover, we already know from item iii) of Proposition 23.3 that $fg \in D(\Delta)$ and $\Delta(fg) = f\,\Delta g + g\,\Delta f + 2\,\nabla f \cdot \nabla g$, in particular $\Delta(fg) \in L^\infty(\mathfrak{m})$. Finally, given that $\nabla f \cdot \nabla g \in W^{1,2}(\mathrm{X})$ by Lemma 28.5 and a polarisation argument, we conclude that $\Delta(fg) \in W^{1,2}(\mathrm{X})$. Hence $fg \in \mathrm{Test}^\infty(\mathrm{X})$, as required. $\square$

We briefly recall the notion of Hessian on a smooth Riemannian manifold (M, g). Given two smooth vector fields X, Y on M, we consider the *covariant derivative* $\nabla_Y X$ *of* X *in the direction of* Y, which is characterised by the following result:

Theorem 28.8 *There exists a unique bilinear map* $(X, Y) \mapsto \nabla_Y X$ *with these properties:*

1) *It is an* affine connection*:*
   - 1a) *It is* tensorial *with respect to* Y*, i.e.* $\nabla_{fY} X = f\,\nabla_Y X$ *holds for all* $f \in C^\infty(M)$ *and* X, Y *smooth vector fields on* M*.*
   - 1b) *It holds that* $\nabla_Y(fX) = Y(f)X + f\,\nabla_Y X$ *for all* $f \in C^\infty(M)$ *and* X, Y *smooth vector fields on* M*.*
2) *It is the* Levi-Civita connection*:*
   - 2a) *It is* torsion-free*, i.e.* $\nabla_X Y - \nabla_Y X = [X, Y]$ *holds for all* X, Y *smooth vector fields on* M*.*
   - 2b) *It is* compatible with the metric*, i.e.* $X\big(\langle Y, Z\rangle\big) = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$ *holds for all* X, Y, Z *smooth vector fields on* M*.*

Proof. The statement follows from the fact that the *Koszul's formula*

$$\langle \nabla_X Y, Z\rangle = X\big(\langle Y, Z\rangle\big) + Y\big(\langle X, Z\rangle\big) - Z\big(\langle X, Y\rangle\big) + \big\langle [X, Y], Z\big\rangle - \big\langle [X, Z], Y\big\rangle - \big\langle [Y, Z], X\big\rangle$$

is equivalent to 1a), 1b), 2a) and 2b). $\square$

Given a smooth vector field X on M, we define the *covariant derivative* ∇X of X as

$$\nabla X(Y,Z) := \langle \nabla_Y X, Z\rangle \quad \text{for all } Y,Z \text{ smooth vector fields on } M. \tag{28.7}$$

Then we define the *Hessian* Hf of a function $f \in C^\infty(M)$ as

$$Hf := \nabla(\nabla f). \tag{28.8}$$

It can be readily proved that the Hessian is a symmetric tensor, i.e.

$$Hf(X,Y) = Hf(Y,X) \quad \text{for all } f \in C^\infty(M) \text{ and } X,Y \text{ smooth vector fields on } M. \tag{28.9}$$

In order to prove it, just observe that item 2b) of Theorem 28.8 yields

$$\begin{aligned} Hf(X,Y) &= \langle \nabla_X \nabla f, Y\rangle = X\big(\langle \nabla f, Y\rangle\big) - \langle \nabla f, \nabla_X Y\rangle = X\big(Y(f)\big) - (\nabla_X Y)(f), \\ Hf(Y,X) &= \langle \nabla_Y \nabla f, X\rangle = Y\big(\langle \nabla f, X\rangle\big) - \langle \nabla f, \nabla_Y X\rangle = Y\big(X(f)\big) - (\nabla_Y X)(f). \end{aligned}$$

By subtracting the second line from the first one, we thus obtain that

$$Hf(X,Y) - Hf(Y,X) = (XY - YX)(f) - \underbrace{(\nabla_X Y - \nabla_Y X)}_{=[X,Y] \text{ by 2a)}}(f) = 0,$$

proving the claim (28.9).

Lemma 28.9 *Let $f \in C^\infty(M)$ be given. Then*

$$\nabla \frac{|\nabla f|^2}{2} = Hf(\nabla f, \cdot). \tag{28.10}$$

Proof. Just observe that for any smooth vector field X on M it holds

$$\Big\langle \nabla \frac{|\nabla f|^2}{2}, X \Big\rangle = \frac{1}{2} X\big(|\nabla f|^2\big) \overset{2b)}{=} \langle \nabla_X \nabla f, \nabla f\rangle = \nabla(\nabla f)(X, \nabla f) \overset{(28.9)}{=} Hf(\nabla f, X),$$

whence the statement follows. □

Remark 28.10 Simple computations show that the identity

$$2\,Hf(\nabla g_1, \nabla g_2) = \nabla(\nabla f \cdot \nabla g_1)\cdot \nabla g_2 + \nabla(\nabla f \cdot \nabla g_2)\cdot \nabla g_1 - \nabla f \cdot \nabla(\nabla g_1 \cdot \nabla g_2) \tag{28.11}$$

is satisfied for every $f, g_1, g_2 \in C^\infty(M)$. ■

29 Lesson [28/02/2018]

In order to introduce the notion of tensor product of Hilbert modules, we first recall what is the tensor product of two Hilbert spaces. Fix H_1, H_2 Hilbert spaces. We call $H_1 \otimes_{\mathrm{Alg}} H_2$ their tensor product as vector spaces, namely the space of formal finite sums $\sum_{i=1}^n v_i \otimes w_i$, with $(v,w) \mapsto v \otimes w$ bilinear. The space $H_1 \otimes_{\mathrm{Alg}} H_2$ satisfies the following universal property:

given any vector space V and any bilinear map $B:\ H_1\times H_2\to V$, there exists a unique linear map $T:\ H_1\otimes_{\mathrm{Alg}} H_2\to V$ such that the diagram

$$\begin{tikzcd} H_1\times H_2 \arrow[r, hook, "\otimes"] \arrow[dr, "B"'] & H_1\otimes_{\mathrm{Alg}} H_2 \arrow[d, "T"] \\ & V \end{tikzcd} \tag{29.1}$$

commutes, where $\otimes:\ H_1\times H_2\hookrightarrow H_1\otimes_{\mathrm{Alg}} H_2$ denotes the map $(v,w)\mapsto v\otimes w$. Hence we can define a scalar product on $H_1\otimes_{\mathrm{Alg}} H_2$ in the following way: first we declare

$$\langle v\otimes w,v'\otimes w'\rangle:=\langle v,v'\rangle_{H_1}\langle w,w'\rangle_{H_2}\qquad\text{for every } v,v'\in H_1 \text{ and } w,w'\in H_2,$$

then we can uniquely extend it to a bilinear operator $\langle\cdot,\cdot\rangle:\ \big[H_1\otimes_{\mathrm{Alg}} H_2\big]^2\to\mathbb{R}$, which is a scalar product as a consequence of the lemma below.

Lemma 29.1 *Let $v_1,\dots,v_n\in H_1$ and $w_1,\dots,w_n\in H_2$ be given. Then*

$$\Big\langle\sum_{i=1}^n v_i\otimes w_i,\sum_{i=1}^n v_i\otimes w_i\Big\rangle\geq 0,$$

with equality if and only if $\sum_{i=1}^n v_i\otimes w_i=0$.

Proof. We can suppose with no loss of generality that H_1 and H_2 are finite-dimensional. Choose orthonormal bases $\mathsf{e}_1,\dots,\mathsf{e}_k$ and $\mathsf{f}_1,\dots,\mathsf{f}_h$ of H_1 and H_2, respectively. Therefore a basis of $H_1\otimes_{\mathrm{Alg}} H_2$ is given by $(\mathsf{e}_i\otimes\mathsf{f}_j)_{i,j}$. Now notice that for any $(a_{ij})_{i,j}\subseteq\mathbb{R}$ it holds

$$\Big\langle\sum_{i,j}a_{ij}\,\mathsf{e}_i\otimes\mathsf{f}_j,\sum_{i,j}a_{ij}\,\mathsf{e}_i\otimes\mathsf{f}_j\Big\rangle=\sum_{i,i',j,j'}a_{ij}\,a_{i'j'}\underbrace{\langle\mathsf{e}_i\otimes\mathsf{f}_j,\mathsf{e}_{i'}\otimes\mathsf{f}_{j'}\rangle}_{=\delta_{(i,j)(i',j')}}=\sum_{i,j}a_{ij}^2,$$

whence the statement follows. □

Then we define the tensor product $H_1\otimes H_2$ of Hilbert spaces as the completion of $H_1\otimes_{\mathrm{Alg}} H_2$ with the respect to the distance coming from $\langle\cdot,\cdot\rangle$.

Now consider two Hilbert modules $\mathscr{H}_1,\mathscr{H}_2$ over a metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m})$. Denote by $\mathscr{H}_1^0,\mathscr{H}_2^0$ the L^0-completions of $\mathscr{H}_1,\mathscr{H}_2$, respectively. Since $\mathscr{H}_1^0,\mathscr{H}_2^0$ are (algebraic) modules over the ring $L^0(\mathfrak{m})$, it makes sense to consider their tensor product $\mathscr{H}_1^0\otimes_{\mathrm{Alg}}\mathscr{H}_2^0$. We endow it with a pointwise scalar product in the following way: first we declare

$$\langle v\otimes v',w\otimes w'\rangle:=\langle v,v'\rangle\langle w,w'\rangle\in L^0(\mathfrak{m})\qquad\text{for every } v,v'\in\mathscr{H}_1^0 \text{ and } w,w'\in\mathscr{H}_2^0,$$

then we can uniquely extend it to an $L^0(\mathfrak{m})$-bilinear operator $\langle\cdot,\cdot\rangle:\ \big[\mathscr{H}_1^0\otimes_{\mathrm{Alg}}\mathscr{H}_2^0\big]^2\to L^0(\mathfrak{m})$. It turns out that such operator is a pointwise scalar product, as we are now going to prove.

Lemma 29.2 *Let $\mathscr{H}^0$ be the L^0-completion of a normed module $\mathscr{H}$. Let $v_1,\dots,v_n\in\mathscr{H}^0$ be given. Then there exist $\mathsf{e}_1,\dots,\mathsf{e}_n\in\mathscr{H}^0$ with the following properties:*

i) $\langle \mathsf{e}_i, \mathsf{e}_j \rangle = 0$ *holds* $\mathfrak{m}$*-a.e. for every* $i \neq j$*,*

ii) $|\mathsf{e}_i| = \chi_{\{|\mathsf{e}_i|>0\}}$ *holds* $\mathfrak{m}$*-a.e. for every* $i = 1, \ldots, n$*,*

iii) *for all* $i = 1, \ldots, n$ *there exist* $(a_{ij})_{j=1}^n \subseteq L^0(\mathfrak{m})$ *such that* $v_i = \sum_{j=1}^n a_{ij}\, \mathsf{e}_j$*.*

Proof. We explicitly build the desired $\mathsf{e}_1, \ldots, \mathsf{e}_n$ by means of a 'Gram-Schmidt orthogonalisation' procedure: we recursively define the e_i's as $\mathsf{e}_1 := \chi_{\{|v_1|>0\}}\, v_1/|v_1|$ and

$$w_k := v_k - \sum_{i=1}^{k-1} \langle v_k, \mathsf{e}_i \rangle\, \mathsf{e}_i, \qquad \mathsf{e}_k := \chi_{\{|w_k|>0\}}\, \frac{w_k}{|w_k|} \quad \text{for every } k = 2, \ldots, n.$$

It can be readily checked that $\mathsf{e}_1, \ldots, \mathsf{e}_n$ satisfy the required properties. □

Remark 29.3 Let $(\mathsf{e}_i)_{i=1}^n \subseteq \mathscr{H}^0$ satisfy items i), ii) of Lemma 29.2. Let $v \in \mathscr{H}^0$ be an element of the form $v = \sum_{i=1}^n a_i\, \mathsf{e}_i$, for some $(a_i)_{i=1}^n \subseteq L^0(\mathfrak{m})$. Then it is easy to check that there is a unique choice of $(b_i)_{i=1}^n \subseteq L^0(\mathfrak{m})$ such that

a) $v = \sum_{i=1}^n b_i\, \mathsf{e}_i$,

b) $b_i = 0$ holds $\mathfrak{m}$-a.e. on $\{\mathsf{e}_i = 0\}$ for all $i = 1, \ldots, n$.

Moreover, we have that $|v|^2 = \sum_{i=1}^n |b_i|^2$ is satisfied $\mathfrak{m}$-a.e. on X. ■

Lemma 29.4 *Let* $A \in \mathscr{H}_1^0 \otimes_{\mathrm{Alg}} \mathscr{H}_2^0$ *be given. Then* $\langle A, A \rangle \geq 0$ *holds* $\mathfrak{m}$*-a.e. on* X*. Moreover, we have that* $\langle A, A \rangle = 0$ *holds* $\mathfrak{m}$*-a.e. on some Borel set* $E \subseteq \mathrm{X}$ *if and only if* $\chi_E\, A = 0$*.*

Proof. Say $A = \sum_{i=1}^n v_i \otimes w_i$. Associate $\mathsf{e}_1, \ldots, \mathsf{e}_n \in \mathscr{H}_1^0$ and $\mathsf{f}_1, \ldots, \mathsf{f}_n \in \mathscr{H}_2^0$ to $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$, respectively, as in Lemma 29.2. Let $b_{ij}, c_{ik} \in L^0(\mathfrak{m})$ be as in Remark 29.3, with $v_i = \sum_{j=1}^n b_{ij}\, \mathsf{e}_j$ and $w_i = \sum_{k=1}^n c_{ik}\, \mathsf{f}_k$ for all $i = 1, \ldots, n$. If $a_{jk} := \sum_{i=1}^n b_{ij}\, c_{ik}$ then

$$\langle A, A \rangle = \sum_{j,k=1}^n |a_{jk}|^2\, |\mathsf{e}_j|^2\, |\mathsf{f}_k|^2 \quad \text{holds } \mathfrak{m}\text{-a.e. on X},$$

whence the statement easily follows. □

Accordingly, it makes sense to define the *pointwise Hilbert-Schmidt norm* as

$$|A|_{\mathsf{HS}} := \sqrt{\langle A, A \rangle} \in L^0(\mathfrak{m})^+ \quad \text{for every } A \in \mathscr{H}_1^0 \otimes_{\mathrm{Alg}} \mathscr{H}_2^0.$$

It immediately stems from Lemma 29.4 that $|A|_{\mathsf{HS}} = 0$ holds $\mathfrak{m}$-a.e. on a Borel set $E \subseteq \mathrm{X}$ if and only if $\chi_E A = 0$.

Definition 29.5 (Tensor product of Hilbert modules) *We define* $\mathscr{H}_1 \otimes \mathscr{H}_2$ *as the completion of the space*

$$\big\{ A \in \mathscr{H}_1^0 \otimes_{\mathrm{Alg}} \mathscr{H}_2^0 \;:\; |A|_{\mathsf{HS}} \in L^2(\mathfrak{m}) \big\}$$

with respect to the norm $A \mapsto \sqrt{\int |A|_{\mathsf{HS}}^2 \, \mathrm{d}\mathfrak{m}}$*. It turns out that* $\mathscr{H}_1 \otimes \mathscr{H}_2$ *is a Hilbert module.*

Lemma 29.6 *Let $D_1 \subseteq \mathscr{H}_1$ and $D_2 \subseteq \mathscr{H}_2$ be dense subsets such that $|v|, |w| \in L^\infty(\mathfrak{m})$ for every $v \in D_1$ and $w \in D_2$. Then the set*

$$\tilde{D} := \left\{ \sum_{i=1}^n v_i \otimes w_i \,:\, v_i \in D_1,\, w_i \in D_2 \right\}$$

is dense in $\mathscr{H}_1 \otimes \mathscr{H}_2$. In particular, $\mathscr{H}_1 \otimes \mathscr{H}_2$ is separable as soon as $\mathscr{H}_1, \mathscr{H}_2$ are separable.

Proof. To prove the first part of the statement, it is clearly sufficient to show that

$$v \otimes w \;\text{ is in the closure of } \tilde{D} \quad \text{ for all } v \in \mathscr{H}_1,\, w \in \mathscr{H}_2 \text{ with } v \otimes w \in \mathscr{H}_1 \otimes \mathscr{H}_2. \tag{29.2}$$

First of all, the closure of $\tilde{D}$ contains $\big\{v \otimes w \,:\, v \in \mathscr{H}_1,\, w \in D_2\big\}$: chosen any $(v_n)_n \subseteq D_1$ converging to v, we have that $|v_n \otimes w - v \otimes w|_{\mathsf{HS}} = \big|(v_n - v) \otimes w\big|_{\mathsf{HS}} = |v_n - v||w| \to 0$ in $L^2(\mathfrak{m})$. In a symmetric way, one can prove that the closure of $\tilde{D}$ contains also $\big\{v \otimes w \,:\, v \in D_1,\, w \in \mathscr{H}_2\big\}$. Therefore $\big\{v \otimes w \,:\, v \in \mathscr{H}_1,\, w \in \mathscr{H}_2,\, |w| \in L^\infty(\mathfrak{m})\big\}$ is contained in the closure of $\tilde{D}$: given any $v \in \mathscr{H}_1$, $w \in \mathscr{H}_2$ with $|w| \in L^\infty(\mathfrak{m})$ and a sequence $(v_n)_n \subseteq D_1$ with $v_n \to v$, we have

$$|v_n \otimes w - v \otimes w|_{\mathsf{HS}} \leq |v_n - v||w| \to 0 \quad \text{ in } L^2(\mathfrak{m}).$$

Finally, take any $v \in \mathscr{H}_1$, $w \in \mathscr{H}_2$ such that $v \otimes w \in \mathscr{H}_1 \otimes \mathscr{H}_2$ and define $w_n := \chi_{\{|w| \leq n\}} w \in \mathscr{H}_2$ for all $n \in \mathbb{N}$. Given that $|v \otimes w_n - v \otimes w|_{\mathsf{HS}} = |v||w_n - w| = \chi_{\{|w| > n\}}|v||w|$ holds $\mathfrak{m}$-a.e. on X for any $n \in \mathbb{N}$, by applying the dominated convergence theorem we conclude that $v \otimes w_n \to v \otimes w$. Therefore the claim (29.2) is proved, thus showing the first part of the statement.

The last part of the statement follows by noticing that any separable Hilbert module admits a countable dense subset made of bounded elements. □

Remark 29.7 Given any Hilbert module $\mathscr{H}$, we obtain the *transposition* operator

$$\mathsf{t} : \mathscr{H} \otimes \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$$

by first declaring that $\mathsf{t}(v \otimes w) := w \otimes v \in \mathscr{H}_2^0 \otimes_{\mathrm{Alg}} \mathscr{H}_1^0$ for all $v \in \mathscr{H}_1^0$, $w \in \mathscr{H}_2^0$ and then extending it by linearity and continuity (notice that it preserves the pointwise norm). It turns out that t is a $L^\infty(\mathfrak{m})$-linear map. Since it is also an involution, i.e. $\mathsf{t} \circ \mathsf{t} = \mathrm{id}_{\mathscr{H} \otimes \mathscr{H}}$, we also see that it is an isometric isomorphism of modules. We shall say that $A \in \mathscr{H} \otimes \mathscr{H}$ is *symmetric* provided $A^{\mathsf{t}} := \mathsf{t}(A) = A$. ■

Definition 29.8 *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}(K, \infty)$ space. Then we define*

$$L^2\big((T^*)^{\otimes 2}\mathrm{X}\big) := L^2(T^*\mathrm{X}) \otimes L^2(T^*\mathrm{X}). \tag{29.3}$$

Given any $A \in L^2\big((T^*)^{\otimes 2}\mathrm{X}\big)$, we define

$$A(X, Y) := A(X \otimes Y) \in L^0(\mathfrak{m}) \quad \textit{for every } X, Y \in L^2(T\mathrm{X}), \tag{29.4}$$

where we set $A(X \otimes Y) := \omega(X)\,\eta(Y)$ for $A = \omega \otimes \eta$ and, since $\big|(\omega \otimes \eta)(X \otimes Y)\big| \leq |\omega||\eta||X||Y|$ holds $\mathfrak{m}$-a.e. on X, we can extend it by linearity and continuity.

We point out that

$$\big|A(X,Y)\big| \leq |A|_{\mathsf{HS}}|X||Y| \quad \text{holds } \mathfrak{m}\text{-a.e. on X} \tag{29.5}$$

for every $A \in L^2\big((T^*)^{\otimes 2}\mathrm{X}\big)$ and $X, Y \in L^2(T\mathrm{X})$.

Lemma 29.9 *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K,\infty)$ *space. Then*

$$\Big\{\sum_{i=1}^n h_i \nabla g_i \;:\; h_i, g_i \in \mathrm{Test}^\infty(\mathrm{X})\Big\} \quad \textit{is dense in } L^2(T\mathrm{X}). \tag{29.6}$$

In particular, it holds that

$$\Big\{\sum_{i=1}^n h_i \nabla g_{1,i} \otimes \nabla g_{2,i} \;:\; h_i, g_{1,i}, g_{2,i} \in \mathrm{Test}^\infty(\mathrm{X})\Big\} \quad \textit{is dense in } L^2(T\mathrm{X}) \otimes L^2(T\mathrm{X}). \tag{29.7}$$

Proof. To get (29.6), recall that $\mathrm{Test}^\infty(\mathrm{X})$ is dense in $W^{1,2}(\mathrm{X})$ and weakly* dense in $L^\infty(\mathfrak{m})$. To deduce (29.7) from (29.6), it suffices to apply Lemma 29.4 and Theorem 28.7. □

Having the formula (28.11) in mind, we thus give the following definition:

Definition 29.10 (The space $W^{2,2}(\mathrm{X})$) *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K,\infty)$ *space, with* $K \in \mathbb{R}$. *Let* $f \in W^{1,2}(\mathrm{X})$. *Then we say that* $f \in W^{2,2}(\mathrm{X})$ *provided there exists* $A \in L^2\big((T^*)^{\otimes 2}\mathrm{X}\big)$ *such that for every choice of* $h, g_1, g_2 \in \mathrm{Test}^\infty(\mathrm{X})$ *it holds that*

$$2\int h\, A(\nabla g_1, \nabla g_2)\,\mathrm{d}\mathfrak{m} = -\int \nabla f \cdot \nabla g_1\, \mathrm{div}(h\nabla g_2) + \nabla f \cdot \nabla g_2\, \mathrm{div}(h \nabla g_1) + h\, \nabla f \cdot \nabla(\nabla g_1 \cdot \nabla g_2)\,\mathrm{d}\mathfrak{m}.$$

Such tensor A*, which is uniquely determined by* (29.7)*, will be unambiguously denoted by* Hf *and called* Hessian *of* f*. Moreover, the resulting vector space* $W^{2,2}(\mathrm{X})$ *is naturally endowed with the norm* $\|\cdot\|_{W^{2,2}(\mathrm{X})}$*, defined as*

$$\|f\|_{W^{2,2}(\mathrm{X})} := \sqrt{\|f\|^2_{L^2(\mathfrak{m})} + \|\mathrm{d}f\|^2_{L^2(T^*\mathrm{X})} + \|Hf\|^2_{L^2((T^*)^{\otimes 2}\mathrm{X})}} \quad \textit{for every } f \in W^{2,2}(\mathrm{X}).$$

Theorem 29.11 *The space* $W^{2,2}(\mathrm{X})$ *is a separable Hilbert space and the Hessian is a closed operator, i.e.*

$$\big\{(f, Hf) \;:\; f \in W^{2,2}(\mathrm{X})\big\} \quad \textit{is closed in } W^{1,2}(\mathrm{X}) \times L^2\big((T^*)^{\otimes 2}\mathrm{X}\big). \tag{29.8}$$

Proof. Proving (29.8) amounts to showing that $f \in W^{2,2}(\mathrm{X})$ and $Hf = A$ whenever a sequence $(f_n)_n \subseteq W^{2,2}(\mathrm{X})$ satisfies $f_n \to f$ in $W^{1,2}(\mathrm{X})$ and $Hf_n \to G$ in $L^2\big((T^*)^{\otimes 2}\mathrm{X}\big)$. This can be achieved by writing the integral formula characterising Hf_n and letting $n \to \infty$. Completeness of $W^{2,2}(\mathrm{X})$ is then a direct consequence of (29.8). Finally, we deduce the separability of $W^{2,2}(\mathrm{X})$ from the fact that the operator $f \mapsto (f, \mathrm{d}f, Hf)$ is an isometry from $W^{2,2}(\mathrm{X})$ to the separable space $L^2(\mathfrak{m}) \times L^2(T^*\mathrm{X}) \times L^2\big((T^*)^{\otimes 2}\mathrm{X}\big)$, provided the latter is endowed with the product norm. □

Remark 29.12 In this framework, the Laplacian is not the trace of the Hessian. ■

30 Lesson [05/03/2018]

Definition 30.1 (Measure-valued Laplacian) *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a metric measure space. Let* $f \in W^{1,2}(\mathrm{X})$. *Then we say that* f *has* measure-valued Laplacian, *briefly* $f \in D(\mathbf{\Delta})$, *provided there exists a finite (signed) Radon measure* μ *on* X *such that*

$$\int g \, \mathrm{d}\mu = -\int \nabla g \cdot \nabla f \, \mathrm{d}\mathfrak{m} \quad \text{for every } g \in \mathrm{LIP}_{bs}(\mathrm{X}). \tag{30.1}$$

The measure μ, *which is uniquely determined by the density of* $\mathrm{LIP}_{bs}(\mathrm{X})$ *in* $C_b(\mathrm{X})$, *will be unambiguously denoted by* $\mathbf{\Delta} f$.

It holds that $D(\mathbf{\Delta})$ is a vector space and that $\mathbf{\Delta} : D(\mathbf{\Delta}) \to \big\{\textit{finite Radon measures on } \mathrm{X}\big\}$ is a linear map. Both properties immediately follow from (30.1).

Remark 30.2 Suppose that (X,d) is bounded. Then

$$\mathbf{\Delta} f(\mathrm{X}) = 0 \quad \text{for every } f \in D(\mathbf{\Delta}). \tag{30.2}$$

Indeed, $g \equiv 1$ trivially belongs to $\mathrm{LIP}_{bs}(\mathrm{X})$, whence (30.1) yields $\mathbf{\Delta} f(\mathrm{X}) = \int \mathrm{d}\mathbf{\Delta} f = 0$. ■

Example 30.3 Let $\mathrm{X} := [0,1]$ and $\mathfrak{m} := \mathcal{L}^1|_{[0,1]}$. Then the identity function $f(x) := x$ belongs to $D(\mathbf{\Delta})$ and $\mathbf{\Delta} f = \delta_0 - \delta_1$.

Lemma 30.4 *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be an* $\mathsf{RCD}(K,\infty)$ *space. Then* $\mathrm{LIP}_{bs}(\mathrm{X})$ *is dense in* $W^{1,2}(\mathrm{X})$.

Proof. We already know that $\mathrm{Test}^\infty(\mathrm{X})$ is dense in $W^{1,2}(\mathrm{X})$ (cf. Proposition 28.4). Then it suffices to prove that $\mathrm{LIP}_{bs}(\mathrm{X})$ is $W^{1,2}(\mathrm{X})$-dense in $\mathrm{Test}^\infty(\mathrm{X})$. To this aim, fix $f \in \mathrm{Test}^\infty(\mathrm{X})$ and define $\chi_n := \big(1 - \mathsf{d}(\cdot, B_n(\bar{x}))\big)^+$ for all $n \in \mathbb{N}$, where $\bar{x} \in \mathrm{X}$ is any fixed point. Now let us call $f_n := \chi_n f \in \mathrm{LIP}_{bs}(\mathrm{X})$ for every $n \in \mathbb{N}$. Then the dominated convergence theorem gives

$$\left.\begin{aligned} |f_n - f| &= |1-\chi_n|\,|f| \longrightarrow 0, \\ |\mathrm{d}f_n - \mathrm{d}f| &\leq |1-\chi_n|\,|\mathrm{d}f| + |\mathrm{d}\chi_n|\,|f| \longrightarrow 0, \end{aligned}\right. \qquad \text{in } L^2(\mathfrak{m}),$$

thus proving that $f_n \to f$ in $W^{1,2}(\mathrm{X})$, as required. □

Proposition 30.5 (Compatibility of Δ and $\mathbf{\Delta}$) *The following properties hold:*

i) *Let* $f \in D(\mathbf{\Delta})$ *satisfy* $\mathbf{\Delta} f = \rho\,\mathfrak{m}$ *for some* $\rho \in L^2(\mathfrak{m})$. *Then* $f \in D(\Delta)$ *and* $\Delta f = \rho$.

ii) *Let* $f \in D(\Delta)$ *satisfy* $\Delta f \in L^1(\mathfrak{m})$. *Then* $f \in D(\mathbf{\Delta})$ *and* $\mathbf{\Delta} f = \Delta f\,\mathfrak{m}$.

Proof. i) We know that $\int g\,\rho\,\mathrm{d}\mathfrak{m} = -\int \nabla g \cdot \nabla f \, \mathrm{d}\mathfrak{m}$ holds for every $g \in \mathrm{LIP}_{bs}(\mathrm{X})$, whence also for every $g \in W^{1,2}(\mathrm{X})$ by Lemma 30.4. This proves that $f \in D(\Delta)$ and $\Delta f = \rho$.
ii) Since $\int g \, \mathrm{d}(\Delta f\,\mathfrak{m}) = \int g\,\Delta f\,\mathrm{d}\mathfrak{m} = -\int \nabla g \cdot \nabla f\,\mathrm{d}\mathfrak{m}$ for every $g \in \mathrm{LIP}_{bs}(\mathrm{X}) \subseteq W^{1,2}(\mathrm{X})$, we see that $f \in D(\mathbf{\Delta})$ and $\mathbf{\Delta} f = \Delta f\,\mathfrak{m}$. □

Lemma 30.6 (Good cut-off functions) *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be an* $\mathsf{RCD}(K,\infty)$ *space. Then there exist* $(\chi_n)_n \subseteq \mathrm{Test}^\infty(\mathrm{X})$ *and* $\bar{x} \in \mathrm{X}$, $C>0$ *such that*

i) $\chi_n = 1$ *on* $B_n(\bar{x})$ *for every* $n \in \mathbb{N}$,

ii) $\mathrm{spt}(\chi_n) \subseteq B_{n+1}(\bar{x})$ *for every* $n \in \mathbb{N}$,

iii) $|\chi_n|, |\nabla\chi_n|, |\Delta\chi_n| \le C$ *hold* $\mathfrak{m}$*-a.e. for every* $n \in \mathbb{N}$.

Proposition 30.7 *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a proper* $\mathsf{RCD}(K,\infty)$ *space, i.e. all bounded closed subsets are compact. Let* $f \in W^{1,2}(\mathrm{X}) \cap L^1(\mathfrak{m})$ *and let* μ *be a finite Radon measure on* X *such that*

$$-\int \nabla g\cdot\nabla f\,\mathrm{d}\mathfrak{m} \ge \int g\,\mathrm{d}\mu \qquad \textit{for every } g \in \mathrm{LIP}_{bs}(\mathrm{X})^+. \tag{30.3}$$

Then $f \in D(\mathbf{\Delta})$ *and* $\mathbf{\Delta} f \ge \mu$.

Proof. Fix $\bar{x} \in \mathrm{X}$ and $(\chi_n)_n$ as in Lemma 30.6. Define $V_n := \big\{g \in \mathrm{LIP}(\mathrm{X}) : \mathrm{spt}(g) \subseteq B_n(\bar{x})\big\}$ for all $n \in \mathbb{N}$. Note that $\mathrm{LIP}_{bs}(\mathrm{X}) = \bigcup_n V_n$. We define the linear map $L : \mathrm{LIP}_{bs}(\mathrm{X}) \to \mathbb{R}$ as

$$L(g) := -\int \nabla g\cdot\nabla f\,\mathrm{d}\mathfrak{m} - \int g\,\mathrm{d}\mu \qquad \text{for every } g \in \mathrm{LIP}_{bs}(\mathrm{X}).$$

Note that $L(g) \ge 0$ whenever $g \ge 0$. Given any $n \in \mathbb{N}$ and $g \in V_n$, we have $\|g\|_{L^\infty(\mathfrak{m})}\chi_n \pm g \ge 0$, so that $\pm L(g) \le \|g\|_{L^\infty(\mathfrak{m})} L(\chi_n)$, or equivalently that $\big|L(g)\big| \le \|g\|_{L^\infty(\mathfrak{m})} L(\chi_n)$. This grants that L can be uniquely extended to a linear continuous map $L : C_c(\mathrm{X}) \to \mathbb{R}$ by Lemma 30.4. Since L is positive, by applying the Riesz representation theorem we deduce that there exists a Radon measure $\nu \ge 0$ on X such that $L(g) = \int g\,\mathrm{d}\nu$ for all $g \in C_c(\mathrm{X})$, thus in particular

$$-\int \nabla f\cdot\nabla g\,\mathrm{d}\mathfrak{m} = \int g\,\mathrm{d}(\mu+\nu) \qquad \text{for every } g \in \mathrm{LIP}_{bs}(\mathrm{X}). \tag{30.4}$$

By choosing $g = \chi_n$ in (30.4) and by using the dominated convergence theorem, we see that

$$\left|\int \chi_n\,\mathrm{d}(\mu+\nu)\right| = \left|\int \nabla\chi_n\cdot\nabla f\,\mathrm{d}\mathfrak{m}\right| = \left|\int f\,\Delta\chi_n\,\mathrm{d}\mathfrak{m}\right| \le C\int_{\mathrm{X}\setminus B_n(\bar{x})} |f|\,\mathrm{d}\mathfrak{m} \longrightarrow 0,$$

where the last inequality is granted by item iii) of Lemma 30.6. We thus deduce that

$$\nu(\mathrm{X}) = \lim_{n\to\infty}\int \chi_n\,\mathrm{d}\nu = -\lim_{n\to\infty}\int \chi_n\,\mathrm{d}\mu = -\mu(\mathrm{X}) < +\infty,$$

whence accordingly ν is a finite measure. In particular, one has that $\mu+\nu$ is a finite measure as well, so that (30.4) yields $f \in D(\mathbf{\Delta})$ and $\mathbf{\Delta} f = \mu+\nu \ge \mu$. □

Corollary 30.8 *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a proper* $\mathsf{RCD}(K,\infty)$ *space. Fix* $f \in \mathrm{Test}^\infty(\mathrm{X})$. *Then it holds that* $|\nabla f|^2 \in D(\mathbf{\Delta})$ *and*

$$\mathbf{\Delta}\frac{|\nabla f|^2}{2} \ge \big(\nabla f\cdot\nabla\Delta f + K|\nabla f|^2\big)\mathfrak{m}. \tag{30.5}$$

Proof. Denote by μ the right hand side of (30.5). We know from (27.4) that

$$-\int \nabla g \cdot \nabla\left(\frac{|\nabla f|^2}{2}\right) \mathrm{d}\mathfrak{m} = \int \Delta g \, \frac{|\nabla f|^2}{2} \, \mathrm{d}\mathfrak{m} \geq \int g \, \mathrm{d}\mu \quad \text{for every } g \in \mathrm{Test}^\infty(\mathrm{X})^+.$$

By regularisation via the mollified heat flow (cf. Proposition 26.8), we see that the previous inequality is verified for every $g \in \mathrm{LIP}_{bs}(\mathrm{X})^+$, so that Proposition 30.7 gives the thesis. $\square$

Given any $f_1, f_2 \in \mathrm{Test}^\infty(\mathrm{X})$, let us define

$$\Gamma_2(f_1, f_2) := \frac{1}{2}\Big[\mathbf{\Delta}(\nabla f_1 \cdot \nabla f_2) - \big(\nabla f_1 \cdot \nabla \Delta f_2 + \nabla f_2 \cdot \nabla \Delta f_1\big)\mathfrak{m}\Big]. \tag{30.6}$$

Notice that $\Gamma_2(f_1, f_2)$ is a finite Radon measure on X and that Γ_2 is bilinear. Then the inequality (30.5) can be restated in the following compact form:

$$\Gamma_2(f, f) \geq K|\nabla f|^2 \mathfrak{m} \quad \text{for every } f \in \mathrm{Test}^\infty(\mathrm{X}). \tag{30.7}$$

Moreover, given any $f, g, h \in \mathrm{Test}^\infty(\mathrm{X})$ we define

$$[\mathrm{H}f](g, h) := \frac{1}{2}\big(\nabla(\nabla f \cdot \nabla g) \cdot \nabla h + \nabla(\nabla f \cdot \nabla h) \cdot \nabla g - \nabla f \cdot \nabla(\nabla g \cdot \nabla h)\big). \tag{30.8}$$

It turns out that $(f, g, h) \mapsto [\mathrm{H}f](g, h)$ is a trilinear map.

The following fundamental result will be proved in the next lesson:

Theorem 30.9 (Key lemma) *Let $f_i, g_i, h_j \in \mathrm{Test}^\infty(\mathrm{X})$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. We define the Radon measure μ on X as*

$$\begin{aligned}\mu := &\sum_{i,i'} g_i \, g_{i'} \left(\Gamma_2(f_i, f_{i'}) - K \langle \nabla f_i, \nabla f_{i'} \rangle \, \mathfrak{m}\right) + \sum_{i,i'} 2 \, g_i \, [\mathrm{H}f_i](f_{i'}, g_{i'}) \, \mathfrak{m} \\ &+ \sum_{i,i'} \frac{\langle \nabla f_i, \nabla f_{i'} \rangle \langle \nabla g_i, \nabla g_{i'} \rangle + \langle \nabla f_i, \nabla g_{i'} \rangle \langle \nabla f_{i'}, \nabla g_i \rangle}{2} \, \mathfrak{m}.\end{aligned} \tag{30.9}$$

Let us write $\mu = \rho \, \mathfrak{m} + \mu^s$, with $\mu^s \perp \mathfrak{m}$. Then $\mu^s \geq 0$ and

$$\left|\sum_{i,j} \langle \nabla f_i, \nabla h_j \rangle \langle \nabla g_i, \nabla h_j \rangle + g_i \, [\mathrm{H}f_i](h_j, h_j)\right|^2 \leq \rho \sum_{j,j'} \big|\langle \nabla h_j, \nabla h_{j'} \rangle\big|^2 \quad \mathfrak{m}\text{-a.e.}. \tag{30.10}$$

Remark 30.10 Consider the case in which $n = 1$ and X is compact, so that the choice $g \equiv 1$ is allowed. Then $\mu = \Gamma_2(f, f) - K|\nabla f|^2\mathfrak{m}$ and (30.10) reads as

$$\rho \left|\sum_j h_j \otimes h_j\right|^2_{\mathsf{HS}} \geq \left|\sum_j [\mathrm{H}f](h_j, h_j)\right|^2 \quad \mathfrak{m}\text{-a.e.}.$$

This shows that Theorem 30.9 can be used, for instance, to show that any test function belongs to the space $W^{2,2}(\mathrm{X})$. ■

31 Lesson [12/03/2018]

Given two non-negative Radon measures μ, ν on X, we define the Radon measure $\sqrt{\mu\nu}$ as

$$\sqrt{\mu\nu} := \sqrt{\frac{\mathrm{d}\mu}{\mathrm{d}\sigma}\frac{\mathrm{d}\nu}{\mathrm{d}\sigma}}\,\sigma \qquad \text{for any Radon measure } \sigma \geq 0 \text{ with } \mu, \nu \ll \sigma. \tag{31.1}$$

Its well-posedness stems from the fact that the function $(a, b) \mapsto \sqrt{ab}$ is 1-homogeneous.

Lemma 31.1 *Let μ_1, μ_2, μ_3 be (finite) Radon measures on* X*. Assume $\lambda^2\mu_1 + 2\lambda\mu_2 + \mu_3 \geq 0$ for every $\lambda \in \mathbb{R}$. Then $\mu_1, \mu_3 \geq 0$ and $\mu_2 \leq \sqrt{\mu_1\mu_3}$.*

Proof. By choosing $\lambda = 0$ we see that $\mu_3 \geq 0$. Given any Borel set $E \subseteq$ X and $\lambda > 0$, we have that $\mu_1(E)+2\mu_2(E)/\lambda+\mu_3(E)/\lambda^2 \geq 0$, so that $\mu_1(E) \geq -\lim_{\lambda\to+\infty} 2\mu_2(E)/\lambda+\mu_3(E)/\lambda^2 = 0$, which shows that $\mu_1 \geq 0$. Now take any Radon measure $\nu \geq 0$ such that $\mu_1, \mu_2, \mu_3 \ll \nu$. Write $\mu_i = f_i\,\nu$ for $i = 1, 2, 3$. Then $\lambda^2 f_1 + 2\lambda f_2 + f_3 \geq 0$ holds ν-a.e., whence accordingly we have that the inequality $f_2 \leq \sqrt{f_1 f_3}$ holds ν-a.e. as well, concluding the proof. $\square$

Lemma 31.2 *Let $n \in \mathbb{N}$ and let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a polynomial with no constant term. Let us fix $f_1, \ldots, f_n \in \mathrm{Test}^\infty(\mathrm{X})$, briefly $\boldsymbol{f} = (f_1, \ldots, f_n)$. Denote by Φ_i the partial derivative of Φ with respect to its i^{th}-entry. Then $\Phi(\boldsymbol{f}) \in \mathrm{Test}^\infty(\mathrm{X})$ and*

$$\Gamma_2\big(\Phi(\boldsymbol{f}), \Phi(\boldsymbol{f})\big) = A + (B + C)\,\mathfrak{m}, \qquad \big|\nabla\Phi(\boldsymbol{f})\big|^2 = D, \tag{31.2}$$

where we set

$$\begin{aligned} A &:= \sum_{i,j=1}^{n} \Phi_i(\boldsymbol{f})\,\Phi_j(\boldsymbol{f})\,\Gamma_2(f_i, f_j), \\ B &:= 2\sum_{i,j,k=1}^{n} \Phi_i(\boldsymbol{f})\,\Phi_{jk}(\boldsymbol{f})\,[Hf_i](f_j, f_k), \\ C &:= \sum_{i,j,k,h=1}^{n} \Phi_{ik}(\boldsymbol{f})\,\Phi_{jh}(\boldsymbol{f})\,\langle\nabla f_i, \nabla f_j\rangle\,\langle\nabla f_k, \nabla f_h\rangle, \\ D &:= \sum_{i,j=1}^{n} \Phi_i(\boldsymbol{f})\,\Phi_j(\boldsymbol{f})\,\langle\nabla f_i, \nabla f_j\rangle. \end{aligned} \tag{31.3}$$

Proof. The fact that $\Phi(\boldsymbol{f}) \in \mathrm{Test}^\infty(\mathrm{X})$ follows from Theorem 28.7. To prove that (31.2) is satisfied it suffices to manipulate the calculus rules described so far; for instance, it can be readily checked that $\mathrm{d}\Phi(\boldsymbol{f}) = \sum_{i=1}^n \Phi_i(\boldsymbol{f})\,\mathrm{d}f_i$ as a consequence of the Leibniz rule. $\square$

Before proving Theorem 30.9 in full generality, we illustrate the ideas underlying its proof by treating a simpler case (the following approach is due to Bakry):

Proposition 31.3 *Let M be a smooth Riemannian manifold with $\Delta\frac{|\nabla f|^2}{2} \geq \nabla f \cdot \nabla\Delta f$ for every $f \in C^\infty(M)$. Then $\Delta\frac{|\nabla f|^2}{2} \geq \nabla f \cdot \nabla\Delta f + |\mathrm{H}f|^2_{\mathsf{op}}$.*

Proof. Let $\Phi(x_1, x_2) := \lambda x_1 + (x_2 - c)^2 - c^2$ for some $\lambda, c \in \mathbb{R}$. Then Lemma 31.2 yields

$$\begin{aligned} 0 \leq & \Gamma_2\big(\lambda f + (h-c)^2, \lambda f + (h-c)^2\big) \\ = & \lambda^2 \Gamma_2(f,f) + 4\lambda(h-c)\Gamma_2(f,h) + 4(h-c)^2 \Gamma_2(h,h) \\ & + 4\lambda \,\mathrm{H}f(\nabla h, \nabla h) + 8(h-c)\,\mathrm{H}h(\nabla h, \nabla h) + 4|\nabla h|^4. \end{aligned}$$

Since c is arbitrary, we can for every point $x \in M$ choose $c = h(x)$, thus getting that the inequality $\lambda^2 \Gamma_2(f,f) + 4\lambda \,\mathrm{H}f(\nabla h, \nabla h) + 4|\nabla h|^4 \geq 0$ holds for all $\lambda \in \mathbb{R}$, whence accordingly one has $\big|\mathrm{H}f(\nabla h, \nabla h)\big| \leq \sqrt{\Gamma_2(f,f)}\,|\nabla h|^2$. Since $\mathrm{H}f$ is symmetric, for all $x \in M$ we have

$$|\mathrm{H}f|_{\mathsf{op}}(x) = \sup\Big\{\big|\mathrm{H}f(\nabla h, \nabla h)\big| \,:\, h \in C^\infty(M),\ |\nabla h|(x) = 1\Big\} \leq \sqrt{\Gamma_2(f,f)}(x),$$

getting the thesis. □

We now restate Theorem 30.9 and then prove it.

Theorem 31.4 (Key lemma) *Let $f_i, g_i, h_j \in \mathrm{Test}^\infty(\mathrm{X})$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. We define the Radon measure μ on X as*

$$\begin{aligned} \mu := & \sum_{i,i'} g_i\, g_{i'} \big(\Gamma_2(f_i, f_{i'}) - K \langle \nabla f_i, \nabla f_{i'}\rangle \,\mathfrak{m}\big) + \sum_{i,i'} 2\, g_i\, [\mathrm{H}f_i](f_{i'}, g_{i'})\,\mathfrak{m} \\ & + \sum_{i,i'} \frac{\langle \nabla f_i, \nabla f_{i'}\rangle \langle \nabla g_i, \nabla g_{i'}\rangle + \langle \nabla f_i, \nabla g_{i'}\rangle \langle \nabla f_{i'}, \nabla g_i\rangle}{2}\,\mathfrak{m}. \end{aligned} \tag{31.4}$$

Let us write $\mu = \rho\,\mathfrak{m} + \mu^s$, with $\mu^s \perp \mathfrak{m}$. Then $\mu^s \geq 0$ and

$$\Big|\sum_{i,j} \langle \nabla f_i, \nabla h_j\rangle \langle \nabla g_i, \nabla h_j\rangle + g_i\, [\mathrm{H}f_i](h_j, h_j)\Big|^2 \leq \rho \sum_{j,j'} \big|\langle \nabla h_j, \nabla h_{j'}\rangle\big|^2 \quad \mathfrak{m}\text{-}a.e.. \tag{31.5}$$

Proof. Given any $\lambda, a_i, b_i, c_j \in \mathbb{R}$, let us define

$$\Phi(x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_m) := \sum_{i=1}^n (\lambda y_i x_i + a_i x_i - b_i y_i) + \sum_{j=1}^m \big((z_j - c_j)^2 - c_j^2\big).$$

Simple computations show that the only non-vanishing derivatives are

$$\partial_{x_i}\Phi = \lambda y_i + a_i, \quad \partial_{y_i}\Phi = \lambda x_i - b_i, \quad \partial_{x_i y_i}\Phi = \lambda, \quad \partial_{z_j}\Phi = 2(z_j - c_j), \quad \partial_{z_j z_j}\Phi = 2.$$

Let $\boldsymbol{f} := (f_1, \ldots, f_n, g_1, \ldots, g_n, h_1, \ldots, h_m) \in \big[\mathrm{Test}^\infty(X)\big]^{2n+m}$, so that $\Phi(\boldsymbol{f}) \in \mathrm{Test}^\infty(\mathrm{X})$ by Lemma 31.2. Note that $\Gamma_2\big(\Phi(\boldsymbol{f}), \Phi(\boldsymbol{f})\big) \geq K\big|\nabla\Phi(\boldsymbol{f})\big|^2\mathfrak{m}$ by (30.7). Moreover, in this case the objects A, B, C, D defined in Lemma 31.2 read as

$$
\begin{aligned}
A(\lambda,a,b,c) &= \sum_{i,i'}(\lambda g_i+a_i)(\lambda g_{i'}+a_{i'})\Gamma_2(f_i,f_{i'})+o.t.,\\
B(\lambda,a,b,c) &= 4\sum_{i,i'}(\lambda g_i+a_i)\lambda[\mathrm{H}f_i](f_{i'},g_{i'})+4\sum_{i,j}(\lambda g_i+a_i)[\mathrm{H}f_i](h_j,h_j)+o.t.,\\
C(\lambda,a,b,c) &= 2\sum_{i,i'}\lambda^2\big(\langle\nabla f_i,\nabla f_{i'}\rangle\,\langle\nabla g_i,\nabla g_{i'}\rangle+\langle\nabla f_i,\nabla g_{i'}\rangle\,\langle\nabla g_i,\nabla f_{i'}\rangle\big)\\
&\qquad+8\lambda\sum_{i,j}\langle\nabla f_i,\nabla h_j\rangle\,\langle\nabla g_i,\nabla h_j\rangle+4\sum_{j,j'}\big|\langle\nabla h_j,\nabla h_{j'}\rangle\big|^2+o.t.,\\
D(\lambda,a,b,c) &= \sum_{i,i'}(\lambda g_i+a_i)(\lambda g_{i'}+a_{i'})\langle\nabla f_i,\nabla f_{i'}\rangle+o.t.,
\end{aligned}
$$

where each o.t.='other terms' contains either a factor $\lambda f_i - b_i$ or a factor $h_j - c_j$. Therefore Lemma 31.2 grants that for any $\lambda \in \mathbb{R}$, $a, b \in \mathbb{R}^n$ and $c \in \mathbb{R}^m$ we have

$$
A(\lambda,a,b,c)+\big(B(\lambda,a,b,c)+C(\lambda,a,b,c)\big)\,\mathfrak{m}\geq KD(\lambda,a,b,c)\,\mathfrak{m}. \tag{31.6}
$$

Now choose a Radon measure $\sigma \geq 0$ such that $\mathfrak{m}, \Gamma_2(f_i, f_{i'}) \ll \sigma$ for all i, i'. Write $\mathfrak{m} = \eta\,\sigma$. Then property (31.6) gives the σ-a.e. inequality $\frac{\mathrm{d}A}{\mathrm{d}\sigma} + (B+C)\eta \geq KD\,\eta$. Now let us choose a sequence $m \mapsto (E^m_\ell)_\ell$ of Borel partitions of X and uniformly bounded $a^{m\ell}_i, b^{m\ell}_i, c^{m\ell}_j \in \mathbb{R}$ with

$$
\sum_{\ell\in\mathbb{N}}a_i^{m\ell}\,\chi_{E^m_\ell}\xrightarrow{m}\lambda g_i,\qquad\sum_{\ell\in\mathbb{N}}b_i^{m\ell}\,\chi_{E^m_\ell}\xrightarrow{m}\lambda f_i,\qquad\sum_{\ell\in\mathbb{N}}c_j^{m\ell}\,\chi_{E^m_\ell}\xrightarrow{m}h_j
$$

with respect to the strong topology of $L^\infty(\sigma)$, for every i, j. Therefore we deduce that

$$
\begin{aligned}
&\sum_{\ell\in\mathbb{N}}\chi_{E^m_\ell}\left[\frac{\mathrm{d}A(\lambda,a^{m\ell},b^{m\ell},c^{m\ell})}{\mathrm{d}\sigma}+\big(B(\lambda,a^{m\ell},b^{m\ell},c^{m\ell})+C(\lambda,a^{m\ell},b^{m\ell},c^{m\ell})\big)\eta\right]\\
&\geq K\sum_{\ell\in\mathbb{N}}\chi_{E^m_\ell}\,D(\lambda,a^{m\ell},b^{m\ell},c^{m\ell})\eta.
\end{aligned}\tag{31.7}
$$

Since both sides of (31.7) are converging in $L^1(\sigma)$, we conclude that $\lambda^2\mu + 2\lambda F + G \geq 0$ for all $\lambda \in \mathbb{R}$, where μ is defined as in (31.4), while

$$
F:=\sum_{i,j}\langle\nabla f_i,\nabla h_j\rangle\,\langle\nabla g_i,\nabla h_j\rangle\,\mathfrak{m}+g_i\,[\mathrm{H}f_i](h_j,h_j)\,\mathfrak{m},\quad G:=\sum_{j,j'}\big|\langle\nabla h_j,\nabla h_{j'}\rangle\big|^2\mathfrak{m}.
$$

Hence Lemma 31.1 grants that $\mu \geq 0$, so in particular $\mu^s \geq 0$, and that $F \leq \sqrt{(\rho\mathfrak{m})G}$, which is nothing but (31.5). This proves the statement. $\square$

Theorem 31.5 *It holds that* $\mathrm{Test}^\infty(\mathrm{X}) \subseteq W^{2,2}(\mathrm{X})$. *Moreover, if we take* $f \in \mathrm{Test}^\infty(\mathrm{X})$ *and we write* $\Gamma_2(f,f) = \gamma_2\,\mathfrak{m} + \Gamma_2^s$, *then* $\Gamma_2^s \geq 0$ *and for all* $g_1, g_2 \in \mathrm{Test}^\infty(\mathrm{X})$ *we have that*

$$
\left.\begin{aligned}
|\mathrm{H}f|^2_{\mathsf{HS}}&\leq\gamma_2-K\,|\nabla f|^2,\\
\mathrm{H}f(\nabla g_1,\nabla g_2)&=[\mathrm{H}f](g_1,g_2)
\end{aligned}\right\}\quad\textit{hold } \mathfrak{m}\textit{-a.e. in } \mathrm{X}. \tag{31.8}
$$

Proof. Apply Theorem 31.4 with $n = 1$. We thus get the $\mathfrak{m}$-a.e. inequality

$$\begin{aligned}
&\Big|\sum_{j=1}^m \langle\nabla f, \nabla h_j\rangle\,\langle\nabla g, \nabla h_j\rangle + g\,[\mathrm{H}f](h_j, h_j)\Big|^2 \\
&\leq \Big(g^2(\gamma_2 - K\,|\nabla f|^2) + 2\,g\,[\mathrm{H}f](f, g)\Big)\sum_{j,j'=1}^m \big|\langle\nabla h_j, \nabla h_{j'}\rangle\big|^2
\end{aligned} \tag{31.9}$$

for any choice of $f, g, h_1, \ldots, h_m \in \mathrm{Test}^\infty(\mathrm{X})$. Since both sides are $W^{1,2}(\mathrm{X})$-continuous with respect to the entry g, we see that (31.9) is actually verified for any $g \in W^{1,2}(\mathrm{X})$. Then by choosing suitable g's, namely identically equal to 1 on an arbitrarily big ball, we deduce that

$$\begin{aligned}
\Big|\sum_{j=1}^m g_j\,[\mathrm{H}f](h_j, h_j)\Big|^2 &\leq \big(\gamma_2 - K\,|\nabla f|^2\big)\sum_{j,j'=1}^m g_j^2\,\big|\langle\nabla h_j, \nabla h_{j'}\rangle\big|^2 \\
&= \big(\gamma_2 - K\,|\nabla f|^2\big)\Big|\sum_{j=1}^m g_j\,\nabla h_j \otimes \nabla h_j\Big|^2
\end{aligned} \tag{31.10}$$

for all $f, g_1, \ldots, g_m, h_1, \ldots, h_m \in \mathrm{Test}^\infty(\mathrm{X})$. Now note that for $f, g, h, h' \in \mathrm{Test}^\infty(\mathrm{X})$ one has

$$\begin{aligned}
2\,[\mathrm{H}f](h, h') &= [\mathrm{H}f](h + h', h + h') - [\mathrm{H}f](h, h) - [\mathrm{H}f](h', h'), \\
g\,(\nabla h \otimes \nabla h' + \nabla h' \otimes \nabla h) &= g\,\big(\nabla(h + h') \otimes \nabla(h + h') - \nabla h \otimes \nabla h - \nabla h' \otimes \nabla h'\big).
\end{aligned}$$

By combining these two identities with (31.10) and the $\mathfrak{m}$-a.e. inequality $\big|\frac{A+A^{\mathrm{t}}}{2}\big|_{\mathsf{HS}}^2 \leq |A|_{\mathsf{HS}}^2$, which is trivially verified for any $A \in L^2(T^{\otimes 2}\mathrm{X})$, we obtain that

$$\Big|\sum_{j=1}^m g_j\,[\mathrm{H}f](h_j, h'_j)\Big| \leq \sqrt{\gamma_2 - K\,|\nabla f|^2}\,\Big|\sum_{j=1}^m g_j\,\nabla h_j \otimes \nabla h'_j\Big| \tag{31.11}$$

holds $\mathfrak{m}$-a.e. for any $f, g_j, h_j, h'_j \in \mathrm{Test}^\infty(\mathrm{X})$. Define $\mathcal{V} \subseteq L^2(T^{\otimes 2}\mathrm{X})$ as the linear span of the tensors of the form $g\,\nabla h \otimes \nabla h'$, with $g, h, h' \in \mathrm{Test}^\infty(\mathrm{X})$. Then the operator $L : \mathcal{V} \to L^1(\mathfrak{m})$, which is given by

$$L\Big(\sum_{j=1}^m g_j\,\nabla h_j \otimes \nabla h'_j\Big) := \sum_{j=1}^m g_j\,[\mathrm{H}f](h_j, h'_j) \quad \text{for every } \sum_{j=1}^m g_j\,\nabla h_j \otimes \nabla h'_j \in \mathcal{V},$$

is well-defined, linear and continuous by (31.11). Since $\mathcal{V}$ is dense in $L^2(T^{\otimes 2}\mathrm{X})$, there exists a unique linear and continuous extension of L to the whole $L^2(T^{\otimes 2}\mathrm{X})$. Such extension is $L^\infty(\mathfrak{m})$-linear by construction, whence it can be viewed as an element $\overline{A}$ of $L^2\big((T^*)^{\otimes 2}\mathrm{X}\big)$. Notice that (31.11) gives $\big|L(A)\big| \leq \sqrt{\gamma_2 - K\,|\nabla f|^2}\,|A|_{\mathsf{HS}}$ for all $A \in \mathcal{V}$, so that $|L|_{\mathsf{HS}} \leq \sqrt{\gamma_2 - K\,|\nabla f|^2}$ and accordingly $|\overline{A}|_{\mathsf{HS}} \leq \sqrt{\gamma_2 - K\,|\nabla f|^2}$ as well. Finally, for any $g, h \in \mathrm{Test}^\infty(\mathrm{X})$ we have

$$\begin{aligned}
2\int g\,\overline{A}(\nabla h \otimes \nabla h)\,\mathrm{d}\mathfrak{m} &= 2\int L\big(g\,\nabla h \otimes \nabla h\big)\,\mathrm{d}\mathfrak{m} \\
&= \int g\,\big(2\,\nabla(\nabla f \cdot \nabla h) \cdot \nabla h - \nabla f \cdot \nabla|\nabla h|^2\big)\,\mathrm{d}\mathfrak{m} \\
&= -\int \nabla f \cdot \nabla h\,\mathrm{div}(\nabla g \cdot \nabla h) + \nabla f \cdot \nabla|\nabla h|^2\,\mathrm{d}\mathfrak{m}.
\end{aligned}$$

Therefore $f \in W^{2,2}(\mathrm{X})$ and (31.8) can be easily checked to hold true. $\square$

Corollary 31.6 *It holds that $D(\Delta) \subseteq W^{2,2}(\mathrm{X})$. Moreover, we have that*

$$\int |\mathrm{H}f|_{\mathsf{HS}}^2 \,\mathrm{d}\mathfrak{m} \leq \int |\Delta f|^2 - K\,|\nabla f|^2 \,\mathrm{d}\mathfrak{m} \quad \textit{for every } f \in D(\Delta). \tag{31.12}$$

Proof. Formula (31.12) holds for all $f \in \mathrm{Test}^\infty(\mathrm{X})$ as a consequence of Theorem 31.4. The general case $f \in D(\Delta)$ follows by approximating f with a sequence $(f_n)_n \subseteq \mathrm{Test}^\infty(\mathrm{X})$. □

Let us define the space $H^{2,2}(\mathrm{X})$ as the $W^{2,2}(\mathrm{X})$-closure of $\mathrm{Test}^\infty(\mathrm{X})$. An important open problem is the following: is it true that $H^{2,2}(\mathrm{X}) = W^{2,2}(\mathrm{X})$?

32 Lesson [14/03/2018]

Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}(K, \infty)$ space. Consider the functional

$$L^2(\mathfrak{m}) \ni f \longmapsto \begin{cases} \int |\mathrm{H}f|_{\mathsf{HS}}^2 \,\mathrm{d}\mathfrak{m} & \text{if } f \in W^{2,2}(\mathrm{X}), \\ +\infty & \text{otherwise.} \end{cases} \tag{32.1}$$

An open problem is the following: is such functional lower semicontinuous?

It is known that such functional is convex and lower semicontinuous when its domain is replaced by $W^{1,2}(\mathrm{X})$.

Proposition 32.1 (Leibniz rule for H) *Let $f_1, f_2 \in W^{2,2}(\mathrm{X}) \cap \mathrm{LIP}(\mathrm{X}) \cap L^\infty(\mathfrak{m})$ be given. Then $f_1 f_2 \in W^{2,2}(\mathrm{X})$ and*

$$\mathrm{H}(f_1 f_2) = f_1\,\mathrm{H}f_2 + f_2\,\mathrm{H}f_1 + \mathrm{d}f_1 \otimes \mathrm{d}f_2 + \mathrm{d}f_2 \otimes \mathrm{d}f_1. \tag{32.2}$$

Proof. By polarisation, it holds that an element $A \in L^2\big((T^*)^{\otimes 2}\mathrm{X}\big)$ coincides with $\mathrm{H}(f_1 f_2)$ if and only if $A^{\mathsf{t}} = A$ and

$$-\int h\,A(\nabla g, \nabla g)\,\mathrm{d}\mathfrak{m} = \int \nabla(f_1 f_2) \cdot \nabla g\,\mathrm{div}(h\nabla g) + h\,\nabla(f_1 f_2) \cdot \nabla\frac{|\nabla g|^2}{2}\,\mathrm{d}\mathfrak{m} \tag{32.3}$$

holds for all $g, h \in \mathrm{Test}^\infty(\mathrm{X})$. By using the Leibniz rule for gradients, we see that the right hand side of (32.3) can be rewritten as

$$\int f_1\,\nabla f_2 \cdot \nabla g\,\mathrm{div}(h\nabla g) + f_2\,\nabla f_1 \cdot \nabla g\,\mathrm{div}(h\nabla g) + h f_1\,\nabla f_2 \cdot \nabla\frac{|\nabla g|^2}{2} + h f_2\,\nabla f_1 \cdot \nabla\frac{|\nabla g|^2}{2}\,\mathrm{d}\mathfrak{m}. \tag{32.4}$$

Moreover, since $f_1, f_2 \in W^{2,2}(\mathrm{X}) \cap \mathrm{LIP}(\mathrm{X}) \cap L^\infty(\mathfrak{m})$, we also have that

$$\begin{aligned} \int h f_2\,\mathrm{H}f_1(\nabla g, \nabla g)\,\mathrm{d}\mathfrak{m} &= -\int \nabla f_1 \cdot \nabla g\,\mathrm{div}(h f_2 \nabla g) + h f_2\,\nabla f_1 \cdot \nabla\frac{|\nabla g|^2}{2}\,\mathrm{d}\mathfrak{m}, \\ \int h f_1\,\mathrm{H}f_2(\nabla g, \nabla g)\,\mathrm{d}\mathfrak{m} &= -\int \nabla f_2 \cdot \nabla g\,\mathrm{div}(h f_1 \nabla g) + h f_1\,\nabla f_2 \cdot \nabla\frac{|\nabla g|^2}{2}\,\mathrm{d}\mathfrak{m}. \end{aligned} \tag{32.5}$$

Therefore (32.4) and (32.5) yield (32.3). Since the expression in (32.2) defines a symmetric tensor, the thesis is achieved. □

Proposition 32.2 (Chain rule for H) *Let $f \in W^{2,2}(\mathrm{X}) \cap \mathrm{LIP}(\mathrm{X})$. Suppose $\varphi \in C^{1,1}(\mathbb{R})$ has bounded derivative and satisfies $\varphi(0) = 0$ if $\mathfrak{m}(\mathrm{X}) = \infty$. Then $\varphi \circ f \in W^{2,2}(\mathrm{X})$ and*

$$\mathrm{H}(\varphi \circ f) = \varphi'' \circ f \, \mathrm{d}f \otimes \mathrm{d}f + \varphi' \circ f \, \mathrm{H}f. \tag{32.6}$$

Proof. The statement can be achieved by using the chain rule for gradients. □

Lemma 32.3 *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be infinitesimally Hilbertian. Let $f \in L^2(\mathfrak{m})$. Then $f \in W^{1,2}(\mathrm{X})$ if and only if there exists $\omega \in L^2(T^*\mathrm{X})$ such that*

$$\int f \operatorname{div}(X) \, \mathrm{d}\mathfrak{m} = -\int \omega(X) \, \mathrm{d}\mathfrak{m} \quad \text{for every } X \in D(\operatorname{div}). \tag{32.7}$$

In this case, it holds that $\omega = \mathrm{d}f$. Moreover, if $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is an $\mathsf{RCD}(K, \infty)$ space for some $K \in \mathbb{R}$, then it suffices to check this property for $X = \nabla g$ with $g \in \mathrm{Test}^\infty(\mathrm{X})$.

Proof. Sufficiency follows from the definition of divergence. To prove necessity, let $X := \nabla \mathsf{h}_t f$ for $t > 0$. Notice that $\operatorname{div}(X) = \Delta \mathsf{h}_t f$. Moreover, since the Cheeger energy decreases along the heat flow, it holds that

$$\int |\nabla \mathsf{h}_{t/2} f|^2 \, \mathrm{d}\mathfrak{m} = -\int f \, \Delta \mathsf{h}_t f \, \mathrm{d}\mathfrak{m} = \int \omega(\nabla \mathsf{h}_t f) \, \mathrm{d}\mathfrak{m} \leq \|\omega\|_{L^2(T^*\mathrm{X})} \left(\int |\nabla \mathsf{h}_{t/2} f|^2 \, \mathrm{d}\mathfrak{m} \right)^{1/2},$$

whence accordingly $\int |\nabla \mathsf{h}_{t/2} f|^2 \, \mathrm{d}\mathfrak{m} \leq \int |\omega|^2 \, \mathrm{d}\mathfrak{m}$. Since the Cheeger energy is lower semicontinuous, we conclude that $f \in W^{1,2}(\mathrm{X})$ and $\omega = \mathrm{d}f$. Finally, the last statement follows from a density argument. □

Proposition 32.4 *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}(K, \infty)$ space. Let $f_1, f_2 \in H^{2,2}(\mathrm{X}) \cap \mathrm{LIP}(\mathrm{X})$ be given. Then $\langle \nabla f_1, \nabla f_2 \rangle \in W^{1,2}(\mathrm{X})$ and*

$$\mathrm{d}\langle \nabla f_1, \nabla f_2 \rangle = \mathrm{H}f_1(\nabla f_2, \cdot) + \mathrm{H}f_2(\nabla f_1, \cdot). \tag{32.8}$$

Proof. By polarisation and by density of test functions in $H^{2,2}(\mathrm{X})$, it is sufficient to show that one has $|\nabla f|^2 \in W^{1,2}(\mathrm{X})$ and $\mathrm{d}|\nabla f|^2 = 2\,\mathrm{H}f(\nabla f, \cdot)$ for every $f \in \mathrm{Test}^\infty(\mathrm{X})$. Given that we have $2 \int h \, \mathrm{H}f(\nabla f, \nabla g) \, \mathrm{d}\mathfrak{m} = -\int |\nabla f|^2 \operatorname{div}(h \nabla g) \, \mathrm{d}\mathfrak{m}$ for all $g, h \in \mathrm{Test}^\infty(\mathrm{X})$, we know that

$$\int |\nabla f|^2 \operatorname{div}(\nabla g) \, \mathrm{d}\mathfrak{m} = -2 \int \mathrm{H}f(\nabla f, \nabla g) \, \mathrm{d}\mathfrak{m} \quad \text{for every } g \in \mathrm{Test}^\infty(\mathrm{X}),$$

whence Lemma 32.3 yields $|\nabla f|^2 \in W^{1,2}(\mathrm{X})$ and $\mathrm{d}|\nabla f|^2 = 2\,\mathrm{H}f(\nabla f, \cdot)$, as required. □

Corollary 32.5 (Locality of H) *Let $f, g \in H^{2,2}(\mathrm{X}) \cap \mathrm{LIP}(\mathrm{X})$ be given. Then*

$$\mathrm{H}f = \mathrm{H}g \quad \text{holds } \mathfrak{m}\text{-a.e. on } \{f = g\}. \tag{32.9}$$

Proof. By linearity of H, it suffices to prove that $\mathrm{H}f = 0$ holds $\mathfrak{m}$-a.e. on $\{f = 0\}$. Given any $g \in \mathrm{Test}^\infty(\mathrm{X})$, we know from Proposition 32.4 that $\langle \nabla f, \nabla g \rangle \in W^{1,2}(\mathrm{X})$ and

$$\mathrm{H}f(\nabla g, \cdot) = \mathrm{d}\langle \nabla f, \nabla g \rangle - \mathrm{H}g(\nabla f, \cdot). \tag{32.10}$$

Since $\nabla f = 0$ holds $\mathfrak{m}$-a.e. on $\{f = 0\}$, we see that the right hand side of (32.10) vanishes $\mathfrak{m}$-a.e. on $\{f = 0\}$. Hence $\mathrm{H}f(\nabla g, \cdot) = 0$ $\mathfrak{m}$-a.e. on $\{f = 0\}$ for all $g \in \mathrm{Test}^\infty(\mathrm{X})$, which implies that $\mathrm{H}f = 0$ $\mathfrak{m}$-a.e. on $\{f = 0\}$, proving the statement. □

Lemma 32.6 *Let $\Omega \subseteq \mathrm{X}$ be an open set. Let $E \subseteq \Omega$ be a Borel set of finite $\mathfrak{m}$-measure such that* $\mathrm{dist}(E, \partial\Omega) > 0$. *Then there exists $h \in \mathrm{Test}^\infty(\mathrm{X})$ such that $h = 1$ on E and* $\mathrm{spt}(h) \subseteq \Omega$.

Given a Borel subset E of X, we define its *essential interior* as

$$\mathrm{ess\,int}(E) := \bigcup \big\{\Omega \,:\, \Omega \subseteq \mathrm{X} \text{ open}, \, \mathfrak{m}(\Omega \setminus E) = 0\big\}. \tag{32.11}$$

By using Lemma 32.6, we can prove that functions in $W^{2,2}(\mathrm{X})$ (but not necessarily in $H^{2,2}(\mathrm{X})$) satisfy a weaker form of locality:

Proposition 32.7 *Let $f \in W^{2,2}(\mathrm{X})$ be given. Then $\mathrm{H}f = 0$ holds $\mathfrak{m}$-a.e. on $\mathrm{ess\,int}\big(\{f = 0\}\big)$.*

Proof. Call Ω the essential interior of $\{f = 0\}$. Define $E_n := \big\{x \in \Omega \,:\, \mathsf{d}(x, \partial\Omega) \geq 1/n\big\}$ for any $n \in \mathbb{N}$. Then Lemma 32.6 gives us a sequence $(h_n)_n \subseteq \mathrm{Test}^\infty(\mathrm{X})$ such that $h_n = 1$ on E_n and $\mathrm{spt}(h_n) \subseteq \Omega$ for every $n \in \mathbb{N}$. Given any $g_1, g_2, h \in \mathrm{Test}^\infty(\mathrm{X})$, we have $hh_n \in \mathrm{Test}^\infty(\mathrm{X})$, thus accordingly $\int hh_n \, \mathrm{H}f(\nabla g_1, \nabla g_2) \, \mathrm{d}\mathfrak{m}$ equals

$$-\int \nabla f \cdot \nabla g_1 \, \mathrm{div}(hh_n \nabla g_2) + \nabla f \cdot \nabla g_2 \, \mathrm{div}(hh_n \nabla g_1) + hh_n \, \nabla f \cdot \nabla(\nabla g_1 \cdot \nabla g_2) \, \mathrm{d}\mathfrak{m}. \tag{32.12}$$

Since the expression in (32.12) vanishes as a consequence of the fact that $f = 0$ $\mathfrak{m}$-a.e. on Ω and $h_n = 0$ on $\mathrm{X} \setminus \Omega$, we deduce that $\int hh_n \, \mathrm{H}f(\nabla g_1, \nabla g_2) \, \mathrm{d}\mathfrak{m} = 0$ is verified for any $n \in \mathbb{N}$ and $g_1, g_2, h \in \mathrm{Test}^\infty(\mathrm{X})$. This grants that $\mathrm{H}f = 0$ holds $\mathfrak{m}$-a.e. on Ω, as required. □

33 Lesson [19/03/2018]

On a Riemannian manifold M, we have for any vector field X and any $f, g \in C^\infty(M)$ that

$$\langle \nabla_{\nabla f} X, \nabla g \rangle = \big\langle \nabla \langle X, \nabla g \rangle, \nabla f \big\rangle - \mathrm{H}g(X, \nabla f). \tag{33.1}$$

Such formula motivates the following definition of covariant derivative for RCD spaces.

Definition 33.1 (Covariant derivative) *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}(K, \infty)$ space. Then a vector field $X \in L^2(T\mathrm{X})$ belongs to $W^{1,2}_C(T\mathrm{X})$ provided there exists $T \in L^2(T^{\otimes 2}\mathrm{X})$ such that*

$$\int h \, T : (\nabla f \otimes \nabla g) \, \mathrm{d}\mathfrak{m} = -\int \langle X, \nabla g \rangle \, \mathrm{div}(h \nabla f) + h \, \mathrm{H}g(X, \nabla f) \, \mathrm{d}\mathfrak{m} \tag{33.2}$$

holds for every $f, g, h \in \mathrm{Test}^\infty(\mathrm{X})$. The element T, which is uniquely determined by (33.2), *is called* covariant derivative *of X and denoted by ∇X. The Sobolev norm of X is defined as*

$$\|X\|_{W^{1,2}_C(T\mathrm{X})} := \Big(\|X\|^2_{L^2(T\mathrm{X})} + \|\nabla X\|^2_{L^2(T^{\otimes 2}\mathrm{X})}\Big)^{1/2}. \tag{33.3}$$

It turns out that the operator $\nabla : W^{1,2}_C(T\mathrm{X}) \to L^2(T^{\otimes 2}\mathrm{X})$ is linear.

In the sequel, we shall denote by $\sharp : L^2\big((T^*)^{\otimes 2}\mathrm{X}\big) \to L^2(T^{\otimes 2}\mathrm{X})$ the Riesz isomorphism.

Theorem 33.2 *The following hold:*

i) $W^{1,2}_C(T\mathrm{X})$ *is a separable Hilbert space.*

ii) *The unbounded operator* $\nabla : L^2(T\mathrm{X}) \to L^2(T^{\otimes 2}\mathrm{X})$ *is closed.*

iii) *If* $f \in H^{2,2}(\mathrm{X}) \cap \mathrm{LIP}(\mathrm{X})$, *then* $\nabla f \in W^{1,2}_C(T\mathrm{X})$ *and* $\nabla(\nabla f) = (\mathrm{H}f)^\sharp$.

Proof. ii) Let $(X_n)_n \subseteq W^{1,2}_C(T\mathrm{X})$ satisfies $X_n \to X$ in $L^2(T\mathrm{X})$ and $\nabla X_n \to T$ in $L^2(T^{\otimes 2}\mathrm{X})$. Then by writing equation (33.2) for X_n and letting $n \to \infty$, we conclude that $X \in W^{1,2}_C(T\mathrm{X})$ and $\nabla X = T$. This proves that ∇ is a closed unbounded operator.
i) Separability follows from the following facts: $X \mapsto (X, \nabla X)$ is an isometry from $W^{1,2}_C(T\mathrm{X})$ to $L^2(T\mathrm{X}) \times L^2(T^{\otimes 2}\mathrm{X})$ and the latter space is separable. Moreover, it directly stems from the construction that the norm $\|\cdot\|_{W^{1,2}_C(T\mathrm{X})}$ satisfies the parallegram identity. Finally, the completeness of $W^{1,2}_C(T\mathrm{X})$ is an immediate consequence of ii).
iii) This can be readily checked by direct computations, by using of Proposition 32.4. □

Proposition 33.3 (Leibniz rule) *Let* $X \in W^{1,2}_C(T\mathrm{X}) \cap L^\infty(T\mathrm{X})$ *and* $f \in W^{1,2}(\mathrm{X}) \cap L^\infty(\mathfrak{m})$. *Then* $fX \in W^{1,2}_C(T\mathrm{X})$ *and* $\nabla(fX) = \nabla f \otimes X + f\,\nabla X$.

Simple computations give the previous formula. Define the class of *test vector fields* as

$$\mathrm{TestV}(\mathrm{X}) := \left\{ \sum_{i=1}^n g_i \nabla f_i \ :\ f_i, g_i \in \mathrm{Test}^\infty(\mathrm{X}) \right\}. \tag{33.4}$$

Then we can formulate an important consequence of Proposition 33.3 in the following way:

Corollary 33.4 *It holds that* $\mathrm{TestV}(\mathrm{X}) \subseteq W^{1,2}_C(T\mathrm{X})$. *Given any* $X = \sum_{i=1}^n g_i \nabla f_i$, *we have*

$$\nabla X = \sum_{i=1}^n \nabla g_i \otimes \nabla f_i + g_i\,(\mathrm{H}f_i)^\sharp. \tag{33.5}$$

Definition 33.5 *We define the space* $H^{1,2}_C(T\mathrm{X})$ *as the* $W^{1,2}_C(T\mathrm{X})$*-closure of* $\mathrm{TestV}(\mathrm{X})$.

Given any $X \in W^{1,2}_C(T\mathrm{X})$ and $Z \in L^0(T\mathrm{X})$, we define the vector field $\nabla_Z X \in L^0(T\mathrm{X})$ as the unique element such that

$$\langle \nabla_Z X, Y \rangle = \nabla X(Z, Y) \qquad \text{for every } Y \in L^0(T\mathrm{X}). \tag{33.6}$$

Observe that $\nabla_Z X \in L^2(T\mathrm{X})$ whenever $Z \in L^\infty(T\mathrm{X})$.

Proposition 33.6 (Compatibility with the metric) *Let* $X, Y \in H^{1,2}_C(T\mathrm{X}) \cap L^\infty(T\mathrm{X})$ *be given. Then* $\langle X, Y \rangle \in W^{1,2}(\mathrm{X})$ *and*

$$\mathrm{d}\langle X, Y \rangle(Z) = \langle \nabla_Z X, Y \rangle + \langle X, \nabla_Z Y \rangle \qquad \textit{for every } Z \in L^0(T\mathrm{X}). \tag{33.7}$$

Proof. First of all, the statement can be obtained for $X, Y \in \mathrm{TestV}(\mathrm{X})$, say that $X = g\nabla f$ and $Y = \tilde{g}\nabla\tilde{f}$, by direct computations. The general case follows by approximation. □

Given any $X, Y \in H^{1,2}_C(T\mathrm{X}) \cap L^\infty(\mathrm{X})$ and $f \in W^{1,2}(\mathrm{X})$, we define

$$\begin{aligned} X(f) &:= \nabla f \cdot X = \mathrm{d}f(X), \\ [X,Y] &:= \nabla_X Y - \nabla_Y X. \end{aligned} \tag{33.8}$$

We call $[X,Y]$ the *commutator*, or *Lie brackets*, between X and Y.

Proposition 33.7 (Torsion-free identity) *Let $X, Y \in H^{1,2}_C(T\mathrm{X}) \cap L^\infty(T\mathrm{X})$. Then*

$$X\big(Y(f)\big) - Y\big(X(f)\big) = [X,Y](f) \quad \textit{for every } f \in H^{2,2}(\mathrm{X}) \cap \mathrm{LIP}(\mathrm{X}). \tag{33.9}$$

Proof. Observe that

$$\begin{aligned} \nabla(\nabla f \cdot Y) \cdot X &= \nabla_X(\nabla f) \cdot Y + \nabla f \cdot \nabla_X Y = \mathrm{H}f(X,Y) + \nabla f \cdot \nabla_X Y, \\ \nabla(\nabla f \cdot X) \cdot Y &= \nabla_Y(\nabla f) \cdot X + \nabla f \cdot \nabla_Y X = \mathrm{H}f(Y,X) + \nabla f \cdot \nabla_Y X. \end{aligned} \tag{33.10}$$

Since $\mathrm{H}f$ is symmetric, by subtracting the second equation of (33.10) from the first one we obtain precisely (33.9). □

Remark 33.8 Since $\big\{\mathrm{d}f \,:\, f \in H^{2,2}(\mathrm{X}) \cap \mathrm{LIP}(\mathrm{X})\big\}$ generates the module $L^2(T^*\mathrm{X})$, we deduce that $[X,Y]$ is the unique element satisfying (33.9). ■

We now want to introduce the notion of exterior differential for RCD spaces.
Given a Riemannian manifold M and a smooth k-form ω over M, it is well-known that $\mathrm{d}\omega$ is given by the following formula: given $X_0, \ldots, X_k$ smooth vector fields on M, one has

$$\mathrm{d}\omega(X_0, \ldots, X_k) = \sum_{i=0}^{k} (-1)^i \, X_i\big(\omega(\ldots, \hat{X}_i, \ldots)\big) + \sum_{i<j} (-1)^{i+j} \, \omega\big([X_i, X_j], \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots\big). \tag{33.11}$$

Such formula actually defines a $k+1$-form, because it is alternating, functorial and linear in each entry. In order to mimic this definition in the case of RCD, we first need to define the exterior power of a Hilbert module.

Given any $L^0(\mathfrak{m})$-normed Hilbert module $\mathscr{H}^0$ and some number $k \in \mathbb{N}$, we define the *exterior power* $\Lambda^k \mathscr{H}^0$ as follows: we set $\Lambda^0 \mathscr{H}^0 := L^0(\mathfrak{m})$ and $\Lambda^1 \mathscr{H}^0 := \mathscr{H}^0$, while for $k \geq 2$

$$\Lambda^k \mathscr{H}^0 := (\mathscr{H}^0)^{\otimes k} / \mathsf{V}_k, \qquad \begin{array}{l} \text{where we call } \mathsf{V}_k \text{ the closed subspace generated by} \\ \text{the elements } v_1 \otimes \ldots \otimes v_k, \text{ with } v_1, \ldots, v_k \in \mathscr{H}^0 \\ \text{and } v_i = v_j \text{ for some } i \neq j. \end{array} \tag{33.12}$$

The equivalence class of an element $v_1 \otimes \ldots \otimes v_k$ is denoted by $v_1 \wedge \ldots \wedge v_k$. The pointwise scalar product between any two such elements is given by

$$\langle v_1 \wedge \ldots \wedge v_k, w_1 \wedge \ldots \wedge w_k \rangle(x) = \det\big(\langle v_i, w_j \rangle(x)\big)_{i,j} \quad \text{for } \mathfrak{m}\text{-a.e. } x \in \mathrm{X}, \tag{33.13}$$

up to a factor $k!$.

Definition 33.9 *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K, \infty)$ *space. Then we denote the* k^{th}*-exterior power of the cotangent module* $L^0(T^*\mathrm{X})$ *by*

$$L^0(\Lambda^k T^*\mathrm{X}) := \Lambda^k L^0(T^*\mathrm{X}), \tag{33.14}$$

while we denote by $L^2(\Lambda^k T^*\mathrm{X})$ *the subspace of* $L^0(\Lambda^k T^*\mathrm{X})$ *consisting of those elements having pointwise norm in* $L^2(\mathfrak{m})$*.*

Then formula (33.11) suggests the following definition:

Definition 33.10 (Exterior derivative) *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K, \infty)$ *space and* $k \in \mathbb{N}$*. Then we say that a* k*-form* $\omega \in L^2(\Lambda^k T^*\mathrm{X})$ *belongs to* $W^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{X})$ *provided there exists a* $(k+1)$*-form* $\eta \in L^2(\Lambda^{k+1} T^*\mathrm{X})$ *such that for any* $X_0, \ldots, X_1 \in \mathrm{TestV}(\mathrm{X})$ *it holds*

$$\begin{aligned} \int \eta(X_0, \ldots, X_k)\,\mathrm{d}\mathfrak{m} = & \sum_{i=0}^{k} (-1)^{i+1}\,\omega(\ldots, \hat{X}_i, \ldots)\,\mathrm{div}(X_i)\,\mathrm{d}\mathfrak{m} \\ & + \sum_{i<j} \int (-1)^{i+j}\,\omega\big([X_i, X_j], \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots\big)\,\mathrm{d}\mathfrak{m}. \end{aligned} \tag{33.15}$$

The element η*, which is uniquely determined, is called* exterior differential *of* ω *and denoted by* $\mathrm{d}\omega$*. Its norm is defined as*

$$\|\omega\|_{W^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{X})} := \Big(\|\omega\|^2_{L^2(\Lambda^k T^*\mathrm{X})} + \|\mathrm{d}\omega\|^2_{L^2(\Lambda^{k+1} T^*\mathrm{X})} \Big)^{1/2}. \tag{33.16}$$

Much like in Theorem 33.2, one can prove that $W^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{X})$ is a separable Hilbert space and that the unbounded operator $\mathrm{d} : L^2(\Lambda^k T^*\mathrm{X}) \to L^2(\Lambda^{k+1} T^*\mathrm{X})$ is closed.

Proposition 33.11 *Let* $f_0, \ldots, f_k \in \mathrm{Test}^\infty(\mathrm{X})$*. Then both the elements* $f_0\,\mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_k$ *and* $\mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_k$ *belong to* $W^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{X})$ *and it holds*

$$\begin{aligned} \mathrm{d}(f_0\,\mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_k) &= \mathrm{d}f_0 \wedge \ldots \wedge \mathrm{d}f_k, \\ \mathrm{d}(\mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_k) &= 0. \end{aligned} \tag{33.17}$$

Definition 33.12 *Given any* $k \in \mathbb{N}$*, we define the space of* test k-forms *on* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *as*

$$\mathrm{TestForm}_k(\mathrm{X}) := \textit{linear span of the } f_0\,\mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_k, \textit{ with } f_0, \ldots, f_k \in \mathrm{Test}^\infty(\mathrm{X}). \tag{33.18}$$

It turns out that $\mathrm{TestForm}_k(\mathrm{X})$ *is dense in* $L^2(\Lambda^k T^*\mathrm{X})$ *for all* $k \in \mathbb{N}$*. We define* $H^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{X})$ *as the* $W^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{X})$*-closure of* $\mathrm{TestForm}_k(\mathrm{X})$*.*

Proposition 33.13 *Let* $\omega \in H^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{X})$*. Then* $\mathrm{d}\omega \in H^{1,2}_{\mathrm{d}}(\Lambda^{k+1} T^*\mathrm{X})$ *and* $\mathrm{d}(\mathrm{d}\omega) = 0$*.*

Proof. The statement holds for any test k-form by Proposition 33.11. The general case follows from the closure of the exterior differential. □

Definition 33.14 (Closed/exact forms) *Let $\omega \in H^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{X})$. Then we say that ω is* closed *provided $\mathrm{d}\omega = 0$, while it is said to be* exact *if there exists $\alpha \in H^{1,2}_{\mathrm{d}}(\Lambda^{k-1} T^*\mathrm{X})$ such that $\omega = \mathrm{d}\alpha$. Any exact form is also closed by Proposition 33.13.*

It can be readily checked that the space of all closed k-forms is strongly closed in the space $L^2(\Lambda^k T^*\mathrm{X})$. Accordingly, the closed k-forms, endowed with the $L^2(\Lambda^k T^*\mathrm{X})$-norm, constitute a Hilbert space. In general, the same fails if we replace 'closed k-forms' with 'exact k-forms', but we point out that the $L^2(\Lambda^k T^*\mathrm{X})$-closure of the space of exact k-forms is a Hilbert space.

Definition 33.15 (de Rham cohomology) *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be any $\mathsf{RCD}(K, \infty)$ space. Then the* de Rham cohomology *is the quotient Hilbert space defined as follows:*

$$\mathsf{H}^k_{\mathrm{dR}}(\mathrm{X}) := \frac{\textit{closed } k\textit{-forms}}{L^2(\Lambda^k T^*\mathrm{X})\textit{-closure of exact } k\textit{-forms}}. \tag{33.19}$$

Exercise 33.16 Let H_1, H_2 be Hilbert spaces. Let $\varphi : H_1 \to H_2$ be a linear and continuous operator. Then there exists a unique linear and continuous operator $\Lambda^k\varphi : \Lambda^k H_1 \to \Lambda^k H_2$ such that $\Lambda^k\varphi(v_1 \wedge \ldots \wedge v_k) = \varphi(v_1) \wedge \ldots \wedge \varphi(v_k)$ is satisfied for every $v_1, \ldots, v_k \in H_1$. Prove that $\|\Lambda^k\varphi\|_{\mathsf{op}} \leq \|\varphi\|^k_{\mathsf{op}}$.

Exercise 33.17 Let $(\mathrm{X}, \mathsf{d}_\mathrm{X}, \mathfrak{m}_\mathrm{X})$ and $(\mathrm{Y}, \mathsf{d}_\mathrm{Y}, \mathfrak{m}_\mathrm{Y})$ be infinitesimally Hilbertian metric measure spaces. Let $\varphi : \mathrm{X} \to \mathrm{Y}$ be a map of bounded deformation. Then there exists a unique linear and continuous operator $\varphi^* : L^2(\Lambda^k T^*\mathrm{Y}) \to L^2(\Lambda^k T^*\mathrm{X})$ such that

$$\varphi^*(\omega_1 \wedge \ldots \wedge \omega_k) = (\varphi^*\omega_1) \wedge \ldots \wedge (\varphi^*\omega_k) \quad \text{for every } \omega_1, \ldots, \omega_k \in L^2(\Lambda^k T^*\mathrm{Y}). \tag{33.20}$$

Moreover, $|\varphi^* A| \leq \mathrm{Lip}(\varphi)^k \, |A| \circ \varphi$ holds $\mathfrak{m}_\mathrm{X}$-a.e. for every $A \in L^2(\Lambda^k T^*\mathrm{Y})$.

Proposition 33.18 *Let $(\mathrm{X}, \mathsf{d}_\mathrm{X}, \mathfrak{m}_\mathrm{X})$ and $(\mathrm{Y}, \mathsf{d}_\mathrm{Y}, \mathfrak{m}_\mathrm{Y})$ be $\mathsf{RCD}(K, \infty)$ spaces. Let $\varphi : \mathrm{X} \to \mathrm{Y}$ be a map of bounded deformation and $\omega \in H^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{Y})$. Then $\varphi^*\omega \in H^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{X})$ and it holds that $\varphi^*(\mathrm{d}\omega) = \mathrm{d}(\varphi^*\omega)$.*

Proof. For any test k-form $\omega = f_0 \, \mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_k$, we have that

$$\varphi^*\omega = f_0 \circ \varphi \, (\varphi^*\mathrm{d}f_1) \wedge \ldots \wedge (\varphi^*\mathrm{d}f_k) = f_0 \circ \varphi \, \mathrm{d}(f_1 \circ \varphi) \wedge \ldots \wedge \mathrm{d}(f_k \circ \varphi),$$

whence Proposition 33.11 grants that $\varphi^*(\mathrm{d}\omega) = \mathrm{d}(\varphi^*\omega)$. The general case follows from the closure of the exterior differential by an approximation argument. □

Corollary 33.19 *Let $k \in \mathbb{N}$ be given. Then the map φ^* as in Proposition 33.18 canonically induces a linear and continuous operator from $\mathsf{H}^k_{\mathrm{dR}}(\mathrm{Y})$ to $\mathsf{H}^k_{\mathrm{dR}}(\mathrm{X})$.*

34 Lesson [21/03/2018]

We briefly recall the Hodge theory for smooth Riemannian manifolds. With abuse of notation, we will sometimes identify tangent and cotangent objects, via the musical isomorphisms.

Let (M,g) be a smooth Riemannian manifold. Then for any $k \in \mathbb{N}$ we can define the de Rham cohomology $\mathsf{H}^k_{\mathrm{dR}}(M)$ as the quotient of closed k-forms over exact k-forms. Observe that this construction makes use only of the smooth structure of the manifold M, in other words the metric g plays no role. For brevity, we denote by L^2_k the space of all L^2 k-forms on the manifold M, which is a Hilbert space if endowed with the scalar product induced by g. Then we define $\delta : L^2_{k+1} \to L^2_k$ as the adjoint of the unbounded operator $\mathrm{d} : L^2_k \to L^2_{k+1}$, i.e. satisfying $\int \langle \delta\omega, \eta \rangle_k \,\mathrm{dVol} = \int \langle \omega, \mathrm{d}\eta \rangle_{n+1} \,\mathrm{dVol}$. Observe that $\mathrm{d}^2 = 0$, whence $\delta^2 = 0$ as well.

Given any 1-form ω, it holds that $\delta\omega = -\mathrm{div}(X)$, where the vector field X corresponds to ω via the musical isomorphism.

Definition 34.1 *We define the* Hodge laplacian *as the unbounded operator* $\Delta_{\mathrm{H}} : L^2_k \to L^2_k$, *which is given by*

$$\Delta_{\mathrm{H}}\omega := (\delta\mathrm{d} + \mathrm{d}\delta)\omega = (\mathrm{d} + \delta)^2\omega. \tag{34.1}$$

A k-form ω is said to be coexact *provided there exists $\eta \in L^2_{k+1}$ such that $\omega = \delta\eta$, while it is said to be* harmonic *if $\Delta_{\mathrm{H}}\omega = 0$.*

Remark 34.2 Given any smooth 0-form f, i.e. any smooth function $f \in C^\infty(M)$, it holds that $\Delta f = -\Delta_{\mathrm{H}} f$. Moreover, one has that

$$\int \langle \eta, \Delta_{\mathrm{H}}\omega \rangle_k \,\mathrm{dVol} = \int \langle \mathrm{d}\eta, \mathrm{d}\omega \rangle_{k+1} \,\mathrm{dVol} + \int \langle \delta\eta, \delta\omega \rangle_{k-1} \,\mathrm{dVol} \tag{34.2}$$

is verified for $\eta, \omega \in L^2_k$. ■

The following result is due to W. V. D. Hodge:

Theorem 34.3 *The following hold:*

i) $L^2_k = \{$*exact k-forms*$\} \oplus \{$*coexact k-forms*$\} \oplus \{$*harmonic k-forms*$\}$,

ii) *For any $[\omega] \in \mathsf{H}^k_{\mathrm{dR}}(M)$ there exists a unique $\eta \in [\omega]$ such that $\Delta_{\mathrm{H}}\eta = 0$,*

iii) *One has $\Delta_{\mathrm{H}}\eta = 0$ if and only if $\mathrm{d}\eta = 0$ and $\delta\eta = 0$.*

Proof. iii) If $\mathrm{d}\eta = 0$ and $\delta\eta = 0$, then trivially $\Delta_{\mathrm{H}}\eta = 0$. Conversely, suppose that $\Delta_{\mathrm{H}}\eta = 0$. Then (34.2) yields $0 = \int \langle \eta, \Delta_{\mathrm{H}}\eta \rangle_k \,\mathrm{dVol} = \int |\mathrm{d}\eta|^2 + |\delta\eta|^2 \,\mathrm{dVol}$, whence $\mathrm{d}\eta = 0$ and $\delta\eta = 0$.
i) Let $\omega = \mathrm{d}\omega'$, $\alpha = \delta\alpha'$ and $\Delta_{\mathrm{H}}\eta = 0$. We have $\int \langle \mathrm{d}\omega', \delta\alpha' \rangle_k \,\mathrm{dVol} = \int \langle \mathrm{d}^2\omega', \alpha' \rangle_{k-1} \,\mathrm{dVol} = 0$. Moreover, it holds that

$$\int \langle \mathrm{d}\omega', \eta \rangle_k \,\mathrm{dVol} = \int \langle \omega', \delta\eta \rangle_{k-1} \,\mathrm{dVol} = 0,$$
$$\int \langle \delta\alpha', \eta \rangle_k \,\mathrm{dVol} = \int \langle \alpha', \mathrm{d}\eta \rangle_{k+1} \,\mathrm{dVol} = 0$$

by item iii). Hence exact, coexact and harmonic k-forms are in direct sum. Now let $\omega \in L^2_k$ be fixed. Choose $\omega' \in L^2_{k-1}$ that minimises the quantity $\|\omega - \mathrm{d}\alpha\|_{L^2_k}$ among all $\alpha \in L^2_{k-1}$. Then the Euler-Lagrange equation yields $\int \langle \omega - \mathrm{d}\omega', \mathrm{d}\alpha\rangle_k \,\mathrm{dVol} = 0$ for all $\alpha \in L^2_{k-1}$, whence we have $\delta(\omega - \mathrm{d}\omega') = 0$. Now let $\beta' \in L^2_{k+1}$ be the minimiser of $\|\omega - \delta\alpha'\|_{L^2_k}$ among all $\alpha' \in L^2_{k+1}$. Then the Euler-Lagrange equation yields $\int \langle \omega - \delta\beta', \delta\alpha'\rangle_k \,\mathrm{dVol} = 0$ for all $\alpha' \in L^2_{k+1}$, whence we have $\mathrm{d}(\omega - \delta\beta') = 0$. Therefore we can write ω as

$$\omega = \underbrace{\mathrm{d}\omega'}_{\text{exact}} + \underbrace{\delta\beta'}_{\text{coexact}} + \underbrace{(\omega - \mathrm{d}\omega' - \delta\beta')}_{\text{harmonic}},$$

thus proving that i) holds.

ii) Let ω be a closed k-form. Since the space of closed k-forms is orthogonal to that of coexact k-forms, there exists a unique $\eta \in L^2_k$ harmonic such that $\omega - \eta$ is an exact k-form. Then it holds that $[\eta] = [\omega] \in \mathsf{H}^k_{\mathrm{dR}}(M)$, thus proving ii). □

In the language of Hodge theory, we can state a sharper form of the Bochner inequality:

$$\Delta \frac{|\omega|^2}{2} \geq -\langle \omega, \Delta_{\mathrm{H}}\omega\rangle + K|\omega|^2 \quad \text{for every smooth 1-form } \omega. \tag{34.3}$$

Actually, the Bochner identity can be written as follows:

$$\Delta \frac{|\omega|^2}{2} = |\nabla\omega|^2_{\mathsf{HS}} - \langle \omega, \Delta_{\mathrm{H}}\omega\rangle + \mathrm{Ric}(\omega,\omega) \quad \text{for every smooth 1-form } \omega. \tag{34.4}$$

Moreover, we define the *connection Laplacian* $\Delta_{\mathrm{C}}X$ of a smooth vector field X as

$$\int \langle \Delta_{\mathrm{C}}X, Y\rangle \,\mathrm{dVol} = -\int \nabla X : \nabla Y \,\mathrm{dVol} \quad \text{for every smooth vector field } Y. \tag{34.5}$$

One can prove that $\Delta\big(|X|^2/2\big) = |\nabla X|^2_{\mathsf{HS}} + \langle X, \Delta_{\mathrm{C}}X\rangle$ holds for any smooth vector field X. We also have that

$$\Delta_{\mathrm{C}}X + \Delta_{\mathrm{H}}X = \mathrm{Ric}(X,\cdot) \quad \text{for every smooth vector field } X, \tag{34.6}$$

which is known as the *Weitzenböck identity.*

Theorem 34.4 (Bochner) *Suppose that* $\mathrm{Ric}_M \geq 0$*. Then*

$$\dim \mathsf{H}^1_{\mathrm{dR}}(M) \leq \dim M, \tag{34.7}$$

with equality if and only if M is a flat torus.

Proof. We know from Theorem 34.3 that the dimension of $\mathsf{H}^1_{\mathrm{dR}}(M)$ coincides with that of the space of all harmonic 1-forms. Then fix an harmonic 1-form ω. We thus have that

$$0 = \int \Delta \frac{|\omega|^2}{2} \,\mathrm{dVol} \overset{(34.4)}{\geq} \int |\nabla\omega|^2_{\mathsf{HS}} \,\mathrm{dVol} - \int \langle \omega, \Delta_{\mathrm{H}}\omega\rangle \,\mathrm{dVol} = \int |\nabla\omega|^2_{\mathsf{HS}} \,\mathrm{dVol}.$$

Therefore $\int |\nabla\omega|^2_{\mathsf{HS}} \,\mathrm{dVol} = 0$, so by using the parallel transport we conclude that the dimension of the space of harmonic 1-forms is smaller than or equal to $\dim M$, proving (34.7). We omit the proof of the last part of the statement. □

We now introduce the Hodge theory for RCD spaces. Hereafter, the space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ will be a fixed $\mathsf{RCD}(K, \infty)$ space.

Definition 34.5 (Codifferential) *We denote by $D(\delta)$ the set of all k-forms $\omega \in L^2(\Lambda^k T^*\mathrm{X})$ such that there exists $\eta \in L^2(\Lambda^{k-1} T^*\mathrm{X})$ for which*

$$\int \langle \omega, \mathrm{d}\alpha \rangle \, \mathrm{d}\mathfrak{m} = \int \langle \eta, \alpha \rangle \, \mathrm{d}\mathfrak{m} \quad \text{holds for every } \alpha \in \mathrm{TestForm}_{k-1}(\mathrm{X}). \tag{34.8}$$

The element η, which is uniquely determined, is denoted by $\delta\omega$ and called codifferential *of ω.*

It is easy to see that δ is a closed unbounded operator.

Proposition 34.6 *It holds that $\mathrm{TestForm}_k(\mathrm{X}) \subseteq D(\delta)$ for all $k \in \mathbb{N}$. More explicitly,*

$$\begin{aligned} \delta(\mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_k) = & \sum_{i=1}^{k} (-1)^i \, \Delta f_i \, \mathrm{d}f_1 \wedge \ldots \wedge \widehat{\mathrm{d}f_i} \wedge \ldots \wedge \mathrm{d}f_k \\ & + \sum_{i<j} (-1)^{i+j} \, [\nabla f_i, \nabla f_j] \wedge \ldots \wedge \widehat{\mathrm{d}f_i} \wedge \ldots \wedge \widehat{\mathrm{d}f_j} \wedge \ldots \wedge \mathrm{d}f_k \end{aligned} \tag{34.9}$$

is verified for every $f_1, \ldots, f_k \in \mathrm{Test}^\infty(\mathrm{X})$.

Definition 34.7 *Let us define $W^{1,2}_{\mathrm{H}}(\Lambda^k T^*\mathrm{X}) := W^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{X}) \cap D(\delta)$ for every $k \in \mathbb{N}$. The norm of an element $\omega \in W^{1,2}_{\mathrm{H}}(\Lambda^k T^*\mathrm{X})$ is given by*

$$\|\omega\|_{W^{1,2}_{\mathrm{H}}(\Lambda^k T^*\mathrm{X})} := \Big(\|\omega\|^2_{L^2(\Lambda^k T^*\mathrm{X})} + \|\mathrm{d}\omega\|^2_{L^2(\Lambda^{k+1} T^*\mathrm{X})} + \|\delta\omega\|^2_{L^2(\Lambda^{k-1} T^*\mathrm{X})} \Big)^{1/2}. \tag{34.10}$$

Finally, let us define $H^{1,2}_{\mathrm{H}}(\Lambda^k T^*\mathrm{X})$ as the $W^{1,2}_{\mathrm{H}}(\Lambda^k T^*\mathrm{X})$-closure of $\mathrm{TestForm}_k(\mathrm{X})$.

We have that $W^{1,2}_{\mathrm{H}}(\Lambda^k T^*\mathrm{X})$ and $H^{1,2}_{\mathrm{H}}(\Lambda^k T^*\mathrm{X})$ are separable Hilbert spaces.

Definition 34.8 (Hodge Laplacian) *Let $\omega \in H^{1,2}_{\mathrm{H}}(\Lambda^k T^*\mathrm{X})$. Then we declare $\omega \in D(\Delta_{\mathrm{H}})$ provided there exists $\eta \in L^2(\Lambda^k T^*\mathrm{X})$ such that*

$$\int \langle \eta, \alpha \rangle \, \mathrm{d}\mathfrak{m} = \int \langle \mathrm{d}\omega, \mathrm{d}\alpha \rangle + \langle \delta\omega, \delta\alpha \rangle \, \mathrm{d}\mathfrak{m} \quad \text{for every } \alpha \in \mathrm{TestForm}_k(\mathrm{X}). \tag{34.11}$$

The element η, which is uniquely determined, is denoted by $\Delta_{\mathrm{H}}\omega$ and called Hodge Laplacian*.*

Definition 34.9 (Harmonic k-forms) *Let $k \in \mathbb{N}$. Then we define $\mathrm{Harm}_k(\mathrm{X})$ as the set of all $\omega \in H^{1,2}_{\mathrm{H}}(\Lambda^k T^*\mathrm{X})$ such that $\Delta_{\mathrm{H}}\omega = 0$. The elements of $\mathrm{Harm}_k(\mathrm{X})$ are called* harmonic*.*

Remark 34.10 It holds that Δ_{H} is a closed unbounded operator. Indeed, suppose $\omega_n \to \omega$ and $\Delta_{\mathrm{H}}\omega_n \to \eta$ in $L^2(\Lambda^k T^*\mathrm{X})$. Observe that

$$\sup_{n \in \mathbb{N}} \int |\mathrm{d}\omega_n|^2 + |\delta\omega_n|^2 \, \mathrm{d}\mathfrak{m} = \sup_{n \in \mathbb{N}} \int \langle \omega_n, \Delta_{\mathrm{H}}\omega_n \rangle \, \mathrm{d}\mathfrak{m} < +\infty,$$

whence it easily follows that $\omega \in D(\Delta_{\mathrm{H}})$ and $\eta = \Delta_{\mathrm{H}}\omega$, since d and δ are closed. ■

Corollary 34.11 *The space $\big(\mathrm{Harm}_k(\mathrm{X}), \|\cdot\|_{L^2(\Lambda^k T^*\mathrm{X})}\big)$ is Hilbert.*

Proof. Direct consequence of the closure of Δ_{H}. □

Theorem 34.12 (Hodge theorem for RCD spaces) *Let $k \in \mathbb{N}$ be given. Then the map*

$$\mathrm{Harm}_k(\mathrm{X}) \ni \omega \longmapsto [\omega] \in \mathsf{H}^k_{\mathrm{dR}}(\mathrm{X}) \tag{34.12}$$

is an isomorphism of Hilbert spaces.

Proof. First of all, observe that any element of $\mathrm{Harm}_k(\mathrm{X})$ is a closed k-form. In analogy with item iii) of Theorem 34.3, we also have that for any $\omega \in H^{1,2}_{\mathrm{H}}(\Lambda^k T^*\mathrm{X})$ it holds

$$\omega \in \mathrm{Harm}_k(\mathrm{X}) \quad \Longleftrightarrow \quad \mathrm{d}\omega = 0 \text{ and } \delta\omega = 0. \tag{34.13}$$

Moreover, we recall the following general functional analytic fact:

$$H \text{ Hilbert space, } V \subseteq H \text{ linear subspace} \quad \Longrightarrow \quad \begin{cases} V^\perp \ni \omega \mapsto \omega + \overline{V} \in H/\overline{V} \\ \text{is an isomorphism.} \end{cases} \tag{34.14}$$

Now let us apply (34.14) with $H := \{\text{closed } k\text{-forms}\}$ and $V := \{\text{exact } k\text{-forms}\}$. Since it holds that $V^\perp = \mathrm{Harm}_k(\mathrm{X})$ by (34.13), we get the thesis. □

35 Lesson [26/03/2018]

Remark 35.1 Let us define the energy functional $\mathcal{E}_{\mathrm{H}} : L^2(\Lambda^k T^*\mathrm{X}) \to [0, +\infty]$ as follows:

$$\mathcal{E}_{\mathrm{H}}(\omega) := \begin{cases} \frac{1}{2}\int |\mathrm{d}\omega|^2 + |\delta\omega|^2 \,\mathrm{d}\mathfrak{m} & \text{if } \omega \in H^{1,2}_{\mathrm{H}}(\Lambda^k T^*\mathrm{X}), \\ +\infty & \text{otherwise.} \end{cases} \tag{35.1}$$

Then $\mathcal{E}_{\mathrm{H}}$ is convex and lower semicontinuous. Moreover, we have that $\omega \in D(\Delta_{\mathrm{H}})$ if and only if $\partial^-\mathcal{E}_{\mathrm{H}}(\omega) \neq \emptyset$. In this case, $\Delta_{\mathrm{H}}\omega$ is the only element of $\partial^-\mathcal{E}_{\mathrm{H}}(\omega)$. ■

Definition 35.2 (Heat flow of forms) *Let $\omega \in L^2(\Lambda^k T^*\mathrm{X})$. Then we denote by $t \mapsto \mathsf{h}_{\mathrm{H},t}\omega$ the unique gradient flow of $\mathcal{E}_{\mathrm{H}}$ starting from ω.*

Exercise 35.3 Prove that

$$\mathsf{h}_{\mathrm{H},t}(\mathrm{d}\omega) = \mathrm{d}\mathsf{h}_{\mathrm{H},t}\omega \quad \text{for every } \omega \in W^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{X}) \text{ and } t \geq 0. \tag{35.2}$$

Moreover, an analogous property is satisfied by the codifferential δ.

Given any closed k-form ω, its (unique) harmonic representative is $\lim_{t\to\infty} \mathsf{h}_{\mathrm{H},t}\,\omega$.

Definition 35.4 (Connection Laplacian) *Let $X \in H^{1,2}_{\mathrm{C}}(T\mathrm{X})$ be given. Then we declare that $X \in D(\Delta_{\mathrm{C}})$ provided there exists $X \in L^2(T\mathrm{X})$ such that*

$$\int \langle Z, X\rangle \,\mathrm{d}\mathfrak{m} = -\int \langle \nabla X, \nabla Y\rangle \,\mathrm{d}\mathfrak{m} \quad \textit{for every } Y \in \mathrm{TestV}(\mathrm{X}). \tag{35.3}$$

The element Z is denoted by $\Delta_{\mathrm{C}}X$ and called connection Laplacian *of ω.*

Remark 35.5 We define the *connection energy* $\mathcal{E}_{\mathrm{C}} : L^2(T\mathrm{X}) \to [0, +\infty]$ as

$$\mathcal{E}_{\mathrm{C}}(X) := \begin{cases} \frac{1}{2}\int |\nabla X|^2_{\mathsf{HS}}\,\mathrm{d}\mathfrak{m} & \text{if } X \in H^{1,2}_{\mathrm{C}}(T\mathrm{X}), \\ +\infty & \text{otherwise.} \end{cases} \tag{35.4}$$

Then $\mathcal{E}_{\mathrm{C}}$ is a convex and lower semicontinuous functional. Moreover, we have that $X \in D(\Delta_{\mathrm{X}})$ if and only if $\partial^-\mathcal{E}_{\mathrm{C}}(X) \neq \emptyset$. In this case, $-\Delta_{\mathrm{C}} X$ is the unique element of $\partial^-\mathcal{E}_{\mathrm{C}}(X)$. ■

Proposition 35.6 *Let $X \in D(\Delta_{\mathrm{C}}) \cap L^\infty(T\mathrm{X})$ be given. Then $|X|^2/2 \in D(\boldsymbol{\Delta})$ and*

$$\boldsymbol{\Delta}\frac{|X|^2}{2} = \big(|\nabla X|^2_{\mathsf{HS}} + \langle X, \Delta_{\mathrm{C}} X\rangle\big)\mathfrak{m}. \tag{35.5}$$

Proof. We prove the statement for $X \in H^{1,2}_{\mathrm{C}}(T\mathrm{X}) \cap L^\infty(T\mathrm{X})$. We know that $|X|^2 \in W^{1,2}(\mathrm{X})$ and $\nabla|X|^2 = 2\,\nabla X(\cdot, X)$. Hence the equalities

$$\begin{aligned}
\int f\big(|\nabla X|^2_{\mathsf{HS}} + \langle X, \Delta_{\mathrm{C}} X\rangle\big)\,\mathrm{d}\mathfrak{m} &= \int f\,|\nabla X|^2_{\mathsf{HS}} - \nabla(fX) : \nabla X\,\mathrm{d}\mathfrak{m} \\
&= \int f\,|\nabla X|^2_{\mathsf{HS}} - (f\,\nabla X + \nabla f \otimes \nabla X) : \nabla X\,\mathrm{d}\mathfrak{m} \\
&= -\int \nabla X(\nabla f, X)\,\mathrm{d}\mathfrak{m} \\
&= -\int \nabla f \cdot \nabla\frac{|X|^2}{2}\,\mathrm{d}\mathfrak{m}
\end{aligned}$$

hold for every $f \in \mathrm{LIP}_{bs}(\mathrm{X})$, thus obtaining (35.5). □

Definition 35.7 (Heat flow of vector fields) *Let $X \in L^2(T\mathrm{X})$ be given. Then we denote by $t \mapsto \mathsf{h}_{\mathrm{C},t} X$ the unique gradient flow of $\mathcal{E}_{\mathrm{C}}$ starting from X.*

Proposition 35.8 *Let $X \in L^2(T\mathrm{X})$. Then it holds that*

$$|\mathsf{h}_{\mathrm{C},t} X|^2 \leq \mathsf{h}_t\big(|X|^2\big) \quad \mathfrak{m}\text{-a.e.} \quad \text{for every } t \geq 0. \tag{35.6}$$

Proof. Fix $t > 0$ and set $F_s := \mathsf{h}_s\big(|\mathsf{h}_{\mathrm{C},t-s} X|^2\big)$ for all $s \in [0, t]$. Then for a.e. $s \in [0, t]$ one has

$$F'_s = \mathsf{h}_s\big(\Delta|\mathsf{h}_{\mathrm{C},t-s} X|^2 - 2\,\langle \mathsf{h}_{\mathrm{C},t-s} X, \Delta_{\mathrm{C}} \mathsf{h}_{\mathrm{C},t-s} X\rangle\big) = \mathsf{h}_s\big(|\nabla \mathsf{h}_{\mathrm{C},t-s} X|^2\big) \geq 0,$$

whence (35.6) immediately follows. □

With the terminology introduced so far, we can restate Theorem 30.9 as follows:

$$\frac{|X|^2}{2} \in D(\boldsymbol{\Delta}) \quad \text{and} \quad \boldsymbol{\Delta}\frac{|X|^2}{2} \geq \big(|\nabla X|^2_{\mathsf{HS}} - \langle X, \Delta_{\mathrm{H}} X\rangle + K|\nabla|^2\big)\mathfrak{m} \tag{35.7}$$

are verified for every $X \in \mathrm{TestV}(\mathrm{X})$.

Lemma 35.9 *It holds that $H^{1,2}_{\mathrm{H}}(T\mathrm{X}) \subseteq H^{1,2}_{\mathrm{C}}(T\mathrm{X})$. More precisely, we have that*

$$\mathcal{E}_{\mathrm{C}}(X) \leq \mathcal{E}_{\mathrm{H}}(X) - \frac{K}{2}\int |X|^2\,\mathrm{d}\mathfrak{m} \quad \text{for every } X \in H^{1,2}_{\mathrm{H}}(T\mathrm{X}). \tag{35.8}$$

Proof. The statement can be proved by integrating the Bochner inequality (35.7). □

In light of the Bochner identity (27.2), it is natural to give the following definition:

$$\mathbf{Ric}(X,Y) := \mathbf{\Delta}\frac{\langle X,Y\rangle}{2} - \left(\langle\nabla X,\nabla Y\rangle - \frac{\langle X,\Delta_{\mathrm{H}}Y\rangle}{2} - \frac{\langle Y,\Delta_{\mathrm{H}}X\rangle}{2}\right)\mathfrak{m} \tag{35.9}$$

for every $X,Y \in \mathrm{TestV}(\mathrm{X})$. We can thus introduce the *Ricci curvature* operator:

Theorem 35.10 (Ricci curvature) *There exists a unique bilinear and continuous extension of* $\mathbf{Ric}$ *to an operator (still denoted by* $\mathbf{Ric}$*) from* $H^{1,2}_{\mathrm{H}}(T\mathrm{X}) \times H^{1,2}_{\mathrm{H}}(T\mathrm{X})$ *to the space of finite Radon measures on* X. *Moreover, it holds that*

$$\begin{aligned} \mathbf{Ric}(X,X) &\geq K|X|^2\mathfrak{m}, \\ \big\|\mathbf{Ric}(X,Y)\big\|_{\mathsf{TV}} &\leq 2\left(\mathcal{E}_{\mathrm{H}}(X) + K^-\|X\|^2_{L^2(T\mathrm{X})}\right)^{1/2}\left(\mathcal{E}_{\mathrm{H}}(Y) + K^-\|Y\|^2_{L^2(T\mathrm{X})}\right)^{1/2} \\ \mathbf{Ric}(X,Y)(\mathrm{X}) &= \int \langle \mathrm{d}X,\mathrm{d}Y\rangle + \langle\delta X,\delta Y\rangle - \nabla X : \nabla Y\,\mathrm{d}\mathfrak{m} \end{aligned} \tag{35.10}$$

for every $X,Y \in H^{1,2}_{\mathrm{H}}(T\mathrm{X})$.

Proof. The first line and the third line in (35.10) are verified for every $X \in \mathrm{TestV}(\mathrm{X})$ by (35.7) and (35.9). In order to prove the second line (for test vector fields), we first consider the case in which $X = Y$ and $K = 0$: since $\mathbf{Ric}(X,X) \geq 0$, we have that

$$\big\|\mathbf{Ric}(X,X)\big\|_{\mathsf{TV}} = \mathbf{Ric}(X,X)(\mathrm{X}) = 2\,\mathcal{E}_{\mathrm{H}}(X) - 2\,\mathcal{E}_{\mathrm{C}}(X) \leq 2\,\mathcal{E}_{\mathrm{H}}(X),$$

which is precisely the second line in (35.10). Its polarised version – for $X,Y \in \mathrm{TestV}(\mathrm{X})$ – can be achieved by noticing that for all $\lambda \in \mathbb{R}$ one has

$$\lambda^2\,\mathbf{Ric}(X,X) + 2\,\lambda\,\mathbf{Ric}(X,Y) + \mathbf{Ric}(Y,Y) = \mathbf{Ric}(\lambda\,X + Y, \lambda\,X + Y) \geq 0,$$

whence $\big|\mathbf{Ric}(X,Y)\big| \leq \big(\mathbf{Ric}(X,X)\,\mathbf{Ric}(Y,Y)\big)^{1/2}$ by Lemma 31.1 and accordingly

$$\big\|\mathbf{Ric}(X,Y)\big\|_{\mathsf{TV}} \leq \Big(\big\|\mathbf{Ric}(X,X)\big\|_{\mathsf{TV}}\,\big\|\mathbf{Ric}(Y,Y)\big\|_{\mathsf{TV}}\Big)^{1/2},$$

which proves the second in (35.10) for $K = 0$. The general case $K \in \mathbb{R}$ can be shown by repeating the same argument with $\widetilde{\mathbf{Ric}}$ instead of $\mathbf{Ric}$, where we set

$$\widetilde{\mathbf{Ric}}(X,Y) := \mathbf{Ric}(X,Y) - K\langle X,Y\rangle\mathfrak{m} \quad \text{for every } X,Y \in \mathrm{TestV}(\mathrm{X}).$$

Finally, once (35.10) is proven for test vector fields, the full statement easily follows. □

The next result shows that the Ricci curvature is 'tensorial':

Proposition 35.11 *Let* $X,Y \in H^{1,2}_{\mathrm{H}}(T\mathrm{X})$ *and* $f \in \mathrm{Test}^\infty(\mathrm{X})$. *Then* $fX \in H^{1,2}_{\mathrm{H}}(T\mathrm{X})$ *and*

$$\mathbf{Ric}(fX,Y) = f\,\mathbf{Ric}(X,Y). \tag{35.11}$$

Proof. Immediate consequence of the defining property (35.9) of $\mathbf{Ric}$. □

Proposition 35.12 *Let $\omega \in L^2(T^*\mathrm{X})$. Then it holds that*

$$|\mathsf{h}_{\mathrm{H},t}\,\omega|^2 \leq e^{-2Kt}\,\mathsf{h}_t\big(|\omega|^2\big) \quad \mathfrak{m}\text{-}a.e. \quad \text{for every } t \geq 0. \tag{35.12}$$

Proof. Fix $t > 0$ and set $F_s := \mathsf{h}_s\big(|\mathsf{h}_{\mathrm{H},t-s}\,\omega|^2\big)$ for all $s \in [0,t]$. Then for a.e. $s \in [0,t]$ one has

$$F_s' = \mathsf{h}_s\big(\Delta|\mathsf{h}_{\mathrm{H},t-s}\,\omega|^2 + 2\,\langle \mathsf{h}_{\mathrm{H},t-s}\,\omega, \Delta_{\mathrm{H}}\mathsf{h}_{\mathrm{H},t-s}\,\omega\rangle\big) \geq 2\,\mathsf{h}_s\big(K|\mathsf{h}_{\mathrm{H},t-s}\,\omega|^2\big),$$

i.e. $F_s' \geq 2KF_s$ for a.e. $s \in [0,1]$. Then (35.12) follows by Gronwall lemma. $\square$