In memory of Boris Dubrovin

by Tamara Grava, Marco Bertola and Davide Guzzetti
SISSA, via Bonomea 265, 34100, Trieste, Italy

Our first acquaintance with Boris Dubrovin was during the academic years '94/'95 and '95/96' when we were accepted as Ph.D. students at SISSA under his supervision. We were among his first Italian students together with Monica Ugaglia and Marta Mazzocco after he moved to Italy in 1992 and became distinguished professor of Mathematical Physics at SISSA in 1993. Boris was preceded by his reputation not only amongst researchers but also undergraduate students thanks to the widespread diffusion of the books "Modern Geometry" (with S.P.Novikov and A.Fomenko). As students it was easy to be intimidated by him at first. During conferences and summer schools, which we often attended in tow, the more informal environment of communal interactions during lunches and dinners allowed us to gain access to his more relaxed, joyful and witty side. This has always been his style: he put the same intensity and focus both in his work and non-academic life, where he encouraged people around him, including ourselves, to enjoy every moment of life.

Boris was a great mentor and when we became faculty members at SISSA he showed us how to be a great colleague and a good friend as well.

His departure has been a great loss for the mathematical physics group at SISSA and the mathematical physics community at large.

Boris has made several groundbreaking contributions to mathematical physics, with far-reaching influences in various areas of mathematics. Possibly one of his best known inventions was the definition of Frobenius manifolds (which we ought now to refer to as "Dubrovin-Frobenius"); their formalization built a surprising bridge between mathematical physics and differential, algebraic and symplectic geometry. The notion of Dubrovin-Frobenius manifold had been introduced by Boris at the beginning of the 90s in order to provide a proper geometric environment and intrinsic formulation of the Witten-Dijkgraaf-Verlinde-Verlinde equations of associativity, which play a central role in conformal field theory. Dubrovin-Frobenius manifolds are flat Riemannian manifolds endowed with a compatible structure of associative Abelian algebra on the tangent space, namely a structure of Frobenius algebra. The simplest spaces that have an algebra structure on their tangent space, are Lie groups. Unlike Lie groups, the structure constants of the algebra in the tangent space of a Dubrovin-Frobenius manifolds are highly nonlinear functions. The axioms introduced by Dubrovin capture, in a comparatively simple differential geometric way, properties of certain important families of 2D topological field theories, and they proved to have a profound connection to the Gromov–Witten theory. The founding work of this theory is the famed "Geometry of 2-D topological field theories" whose numerous pages all his student (and undoubtedly countless others) have religiously traversed.

In his early works Boris showed that there is a one to one correspondence between semi-simple Dubrovin-Frobenius manifolds and integrable systems of hydrodynamic type that were classified in his earlier works with S.P. Novikov. Then in a series of works Boris and Youjin Zhang showed, using the theory of Dubrovin-Frobenius manifolds, how to associate an integrable dispersive PDEs to integrable systems of hydrodynamic type and then they showed that a particular solution to those PDEs matches the generating function of the intersection numbers of a given cohomological field theories (CohFTs). Indeed Dubrovin-Frobenius manifolds encode the genus zero part of general cohomological field theories (CohFTs). One of these work is the gargantuan

186-page "Normal forms of hierarchies of integrable PDEs, Dubrovin-Frobenius manifolds and Gromov - Witten invariants" (arXiv:math/0108160) in collaboration with Youjin Zhang, which has been subject to many years of refinement but never submitted in final form to a journal.

This very general theory contains, as a particular case, the celebrated Witten-Kontsevich KdV construction for the intersection numbers on the moduli spaces of stable algebraic curves which is also called the Gromov-Witten theory of a point. Another important application of the Dubrovin-Zhang approach to CohFT is the completion of the Gromov–Witten invariants of the complex projective line in terms of a new integrable hierarchy, the so-called extended Toda lattice (joint with G. Carlet and Y. Zhang). This work completed the analysis initiated earlier by T.Eguchi et all and by A. Okounkov and R. Pandharipande. Very recently Boris in collaboration with D. Yang and M.Bertola, developed the concept of topological ODEs to compute in a simple but very efficient way correlators of cohomological field theories and random matrices.

Another prominent example of Frobenius manifold is quantum cohomology, defined by a certain deformation of the classical cohomology, related to Gromov–Witten invariants. In 1998 Dubrovin formulated a conjecture expressing the Stokes and central connection matrices of the quantum cohomology of Fano varieties in terms of full exceptional collections in the derived categories of coherent sheaves on the variety. The conjecture has been refined and verified by several authors in important cases, including Dubrovin himself (joint with G. Cotti and D. Guzzetti). It may be considered part of Kontsevich's Homological Mirror Symmetry.

Dubrovin-Frobenius manifolds play a central role in the mathematical formulation of mirror symmetry, one of the most influential ideas brought to mathematics by theoretical physics. In a nutshell, mirror symmetry is the subtle and deep relation between two mathematical objects: on one side of the mirror is the quantum cohomology of a projective variety, on the other side - the Saito theory of a hypersurface singularity or some more complex Landau-Ginzburg model, both of which produce Dubrovin-Frobenius manifolds. Mirror symmetry can be expressed as the isomorphism between these two Dubrovin-Frobenius manifolds. Witten and others had asked what corresponds, across the mirror, to the Gauss-Manin connection of the hypersurface singularity. Dubrovin's theory provides a natural answer to this question, known as Dubrovin's connection. These and other notions of Dubrovin-Frobenius manifolds are central to various important results, most prominently Givental's mirror theorems.

Dubrovin's approach to integrable systems of PDEs was also central in the development of the double ramification hierarchies. Introduced first by A.Buryak and inspired by Hamiltonian systems arising in Y.Eliashberg, A.Givental and H.Hofer's symplectic field theory, the double ramification hierarchy is a novel construction of integrable systems from intersection theory on the moduli spaces of curves. Dubrovin, together with A.Buryak, J.Guéré and P. Rossi, developed this notion and, in particular, made precise Buryak's early conjectures on a deep and mysterious relation between the double ramification hierachies and the Dubrovin-Zhang hierarchy. In this context the double ramification hierarchy (which provides among other things a general approach to quantization of integrable systems of PDEs) can be seen as the most canonical form possible for a hierarchy of topological type. The equivalence conjecture between the two hierarchies , still open in the general case, has remarkable consequences both from the mathematical physics viewpoint and the topology of the moduli spaces of curves.

Dubrovin-Frobenius manifolds play also an important role in the theory of isomonodromic deformations and in the theory of integrable PDEs. Motivated by the study of algebraic Frobenius manifolds, Boris proposed a new approach to the problem of classification of algebraic solutions of the Schlesinger equations based on an explicit description of the structure of analytic continuation of solutions in terms of the action of braid groups on the space of monodromy data. Based on this approach Boris and M. Mazzocco classified the algebraic solutions of a particular case of the Painlevé-VI equation. Later, the Dubrovin - Mazzocco's approach was extended by O.Lysovyy et al. to a complete classification of algebraic solutions of the general Painlevé-VI equation. Recently, J.Bourgain, A.Gamburd, and P.Sarnak have found a remarkable connection between the theory of an affine sieve for nonlinear actions and the classification of algebraic solutions to the Painlevé-VI equation.

Another important achievement of Boris is related to the universality of critical behaviour of solutions to Hamiltonian PDEs developed together with C. Klein A. Moro and T. Grava. The study of properties of solutions to systems of nonlinear Hamiltonian PDEs with slowly varying initial conditions gave rise to the remarkable discovery of the phenomenon of universality of the behaviour of a generic solution at the point of phase transition from regular to oscillatory behaviour.

After about 25 years since the introduction of the notion of Frobenius manifold by Boris, the citation count on the topic reaches easily many thousands of relevant papers, and this tally is destined to grow.

Finally Boris Dubrovin is among the world top authorities in integrable systems. He is one of the founders of the algebro-geometric approach in the theory of nonlinear PDEs with S.P. Novikov and A. Its and V. Matveev and the geometric theory of Hamiltonian systems of hydrodinamic type with S.P. Novikov.

Boris has always maintained high standard and intensity at work; during the final period of his life, even if afflicted by a progressively debilitating disease, he maintained the same outstanding level of production. In his last series of papers with Si Qui Liu, Yang Di and Youjin Zhang, and Yand Di and Don Zagier, a remarkable connection between intersection theory of Hodge classes on the Deligne–Mumford moduli spaces and integrable systems was established.

For his scientific achievements he received a prize of the Moscow Mathematical Society in 1976, together with A.Its and I.Krichever. He held an invited talk at the International Congress of Mathematical Physicists at Swansea (1988), a plenary talk at the European Congress of Mathematicians at Budapest (1996), an invited talk at the International Congress of Mathematicians in Berlin (1998) and a plenary talk at the International Congress of Mathematical Physicists at Rio de Janeiro (2006).

SISSA and the scientific community at large have lost not only an extraordinary mathematical physicist, but also a great colleague and friend. His enduring scientific legacy is carries on by 25 former Ph.D. students and 15 post-docs who are now working in academia or industry across the world.

He was a generous mentor and inspiring friend. He has been always a very sporty person organizing skiing trips and long walks with the mathematical physics group and his visitors.

His tenacity and courage during the illness have been a source of inspiration and admiration for all the friends and colleagues that have been close to him. Boris is survived by his wife Irina and his two daughters Dasha and Lisa.