SISSA – Mathematics Area

Entrance examination for the course in Mathematical Analysis, Modelling, and Applications

March 10, 2025

Solve FIVE of the following problems. Mark in the table below the exercises you have chosen. These exercises only will be considered for the selection.

Mark here the FIVE problems you have chosen																			
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

Mathematical Analysis

1. Consider the Cauchy problem

$$\dot{x} = f(x)\,, \quad x(0) = x_0\,, \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field of class C^1 satisfying, for some $m \in \mathbb{N}$,

$$|f(x)| \leq |x|^m\,, \quad \forall x \in \mathbb{R}^n\,.$$

Prove that, for any $x_0 \in \mathbb{R}^n$, the solution of (1) is defined on a time interval $[0, T_{\max})$ where

(a) if $m = 1$ then $T_{\max} = +\infty$;

(b) if $m \geq 2$ then there is a constant $c_m > 0$ such that $T_{\max} \geq \frac{c_m}{|x_0|^{m-1}}$, for any $x_0 \neq 0$, and $T_{\max} = +\infty$ if $x_0 = 0$.

2. Prove the following facts:

(a) For any sequence of real numbers $(c_n)_{n\in\mathbb{N}}$ with $c_n \to \infty$, and any set $F \subset \mathbb{R}$ with finite measure it holds

$$\lim_{n\to\infty} \int_F \sin^2(c_n x)dx = \frac{\lambda(F)}{2},$$

where $\lambda(F)$ denotes the Lebesgue measure of F.

(b) Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that

$$f(x) := \lim_{n\to\infty} \sin(\alpha_n x)$$

exists on a set $E \subset \mathbb{R}$ of positive measure. Prove that $(\alpha_n)_{n\in\mathbb{N}}$ has finite limit.

3. Let $K : [0,1] \times [0,1] \to \mathbb{R}$ be a measurable function satisfying

$$M_1 := \sup_{x\in[0,1]} \int_{[0,1]} |K(x,y)|dy < +\infty$$
$$M_2 := \sup_{y\in[0,1]} \int_{[0,1]} |K(x,y)|dx < +\infty\,.$$

Prove that the integral operator $A : L^2([0,1]) \to L^2([0,1])$ defined by

$$(Au)(x) := \int_{[0,1]} K(x,y)\,u(y)\,dy$$

is bounded on $L^2([0,1])$, and

$$\|Au\|_{L^2([0,1])} \leq (M_1 M_2)^{1/2} \|u\|_{L^2([0,1])}\,.$$

4. Let $\ell^2 = \left\{ c = (c_n)_{n\in\mathbb{N}} \mid c_n \in \mathbb{C}\,,\ \sum_{n=0}^{\infty} |c_n|^2 < +\infty \right\}$. For all $c \in \ell^2$, consider the power series

$$f_c(z) := \sum_{n=0}^{\infty} c_n z^n\,.$$

(a) Prove that for any $c \in \ell^2$ the function $f_c : D \to \mathbb{C}$ is holomorphic on the open unit disk $D = \{z \in \mathbb{C} \mid |z| < 1\}$.

(b) Let $(a_k)_{k\in\mathbb{N}}$ be a sequence of complex numbers in D with an accumulation point[1] in D. Consider the following vectors of ℓ^2:

$$h_k = (1, a_k, a_k^2, a_k^3, \dots), \qquad k = 1, 2, \dots\,.$$

Show that $\ell^2 = \overline{\operatorname{span}\{h_1, h_2, \dots\}}$.

5. Consider $T : C([0,1]) \to C([0,1])$ defined by

$$(Tf)(x) = \int_0^{1-x} f(y)dy\,, \quad x \in [0,1]\,.$$

(a) Prove that T is a linear, bounded, compact operator on $C([0,1])$.

(b) Compute the spectrum and the eigenvalues of T.

6. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a scalar function of class C^1 such that

$$\lim_{|x|\to+\infty} f(x) = +\infty\,,$$

with a unique critical point $\underline{x} \in \mathbb{R}^n$. Prove that

(a) for any $x_0 \in \mathbb{R}^n$ the solution $x(t)$ of

$$\dot{x} = -(\nabla f)(x)\,, \quad x(0) = x_0\,,$$

is defined for all $t \geq 0$;

(b) the ω-limit set

$$\omega(x_0) := \big\{ y \in \mathbb{R}^n \mid \text{there exists a sequence } t_n \to +\infty \text{ such that } x(t_n) \to y \big\}$$

is contained in $f^{-1}(\alpha)$ where $\alpha = \inf_{t\geq 0} f(x(t))$;

(c) $\omega(x_0) = \{\underline{x}\}$ and $\lim_{t\to+\infty} x(t) = \underline{x}$.

[1] A point $a \in \mathbb{C}$ is an accumulation point for a sequence $(a_k)_{k\in\mathbb{N}} \subset \mathbb{C}$ if any neighborhood of a contains infinitely many elements of the sequence different from the point a itself.

7. For $s \geq 0$, let $H^s(\mathbb{T})$ be the Hilbert space of Lebesgue-measurable, 2π-periodic functions $f : \mathbb{T} \equiv [-\pi, \pi] \to \mathbb{C}$ of the form

$$f(x) = \sum_{k \in \mathbb{Z}} c_k \, e^{ikx}, \qquad x \in [-\pi, \pi], \tag{2}$$

where c_k are complex numbers such that

$$\|f\|_{H^s} = \left(\sum_{k \in \mathbb{Z}} (1 + |k|^2)^s \, |c_k|^2 \right)^{1/2} < \infty.$$

(a) Show that if $s > 1/2$ then there is a positive constant $C_s > 0$ such that

$$\|f\|_{L^\infty} \leq C_s \, \|f\|_{H^s} \,, \tag{3}$$

and the evaluation functional

$$\mathcal{E} : H^s(\mathbb{T}) \longrightarrow \mathbb{C} \,, \quad f \longmapsto \mathcal{E}(f) = f(0) \,, \tag{4}$$

is continuous.

(b) Prove that if $s = 1/2$ the embedding (3) fails for any constant $C_{1/2}$.

8. Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and recall that a symmetric bounded linear operator A on $\mathcal{H}$ is said to be *positive semi-definite* if $\langle Ax, x \rangle \geq 0$ for any $x \in \mathcal{H}$. Let A and B be two symmetric, bounded linear operators on $\mathcal{H}$. Prove or disprove the following statements:

(a) If $\langle Ax, x \rangle = \langle Bx, x \rangle$ for any $x \in \mathcal{H}$, then $A = B$ on $\mathcal{H}$.

(b) If $A - \mathbb{1}$ is positive semi-definite, where $\mathbb{1}$ denotes the identity on $\mathcal{H}$, then A is invertible and $\mathbb{1} - A^{-1}$ is positive semi-definite.

(c) If A, B and $A - B$ are positive semi-definite, then $A^2 - B^2$ is positive semi-definite.

9. Let $\alpha \in [0, 1]$. Show that all the solutions of

$$\ddot{x} + x - \sin(\alpha x) = 0$$

are periodic.

10. Let X be an infinite–dimensional Banach space and consider the unit sphere:

$$S = \{x \in X \mid \|x\| = 1\}.$$

Prove that the closure $\overline{S}$ with respect to the weak topology $\sigma(X, X^*)$ is the unit ball:

$$B = \{x \in X \mid \|x\| \leq 1\}.$$

Numerical Analysis

11. Consider the one-dimensional partial differential equation

$$\frac{\partial \mathbf{Q}}{\partial t} + A\frac{\partial \mathbf{Q}}{\partial x} = 0, \qquad (x,t) \in (-\infty,+\infty) \times (0,+\infty), \tag{5}$$

$$\mathbf{Q}(x,0) = \mathbf{h}(x), \qquad x \in (-\infty,+\infty), \tag{6}$$

with

$$A = \begin{pmatrix} 1 & 2 & -2 \\ -1 & 1 & 1 \\ -1 & 2 & 0 \end{pmatrix} \in \mathbb{R}^{3\times 3} \quad \text{and} \quad \mathbf{Q}(x,t) \in \mathbb{R}^3. \tag{7}$$

(a) Diagonalise the matrix A (i.e., $A = R\Lambda R^{-1}$).

(b) Write an equation for the variables $\mathbf{C} := R^{-1}\mathbf{Q}$.

(c) Take advantage of the variables $\mathbf{C}$ and the equation associated to them to find a closed-form solution for $\mathbf{Q}(x,t)$.

12. Consider the family of one-dimensional finite difference schemes defined as:

$$\alpha W_{j-1} + W_j + \alpha W_{j+1} = a\left(\frac{U_{j+1} - U_{j-1}}{2h}\right) + b\left(\frac{U_{j+2} - U_{j-2}}{4h}\right), \tag{8}$$

where $U_j \approx u(x_j)$ and $W_j \approx u'(x_j)$ are respectively the approximation of the function $u(x)$ and its derivative $u'(x)$ at the point x_j.

(a) Suppose now that $\alpha = 0$. Find the coefficients a and b so that the scheme in equation (8) is fourth-order accurate (*Hint: consider the Taylor expansion of* $u(x_{j+2})$, $u(x_{j-2})$, $u(x_{j+1})$ *and* $u(x_{j-1})$ *in* x_j).

(b) Consider now the case for $\alpha \neq 0$. Find again the coefficients a and b so that the scheme in equation (8) is fourth-order accurate (*Hint: consider the Taylor expansion of* $u'(x_{j+1})$ *and* $u'(x_{j-1})$ *in* x_j).

(c) Based on the family of numerical schemes obtained in the previous point, is it possible to write an equation that only depends on quantities defined on the stencil $(j-1, j, j+1)$?

(d) In point (1) and in point (3) two different schemes were obtained, both fourth-order accurate: how do we obtain W_j in the two schemes? Which one is more computationally expensive?

13. Let $A \in \mathbb{R}^{n\times n}$, non-singular (invertible) matrix. Let $u, v \in \mathbb{R}^n$; we define the rank-1 perturbation of A as $\widehat{A} = A + uv^T$. The Sherman-Morrison formula provides the following expression for $\widehat{A}^{-1}$, if it exists:

$$\widehat{A}^{-1} = \left(A + uv^T\right)^{-1} = A^{-1} - \left(\frac{1}{s_A}\right) A^{-1}uv^TA^{-1} \quad \text{with} \quad s_A = 1 + v^TA^{-1}u$$

(a) Consider the system $\left(A + uv^T\right)x = b$, and define $y = v^Tx$. Write down the $(n+1)\times(n+1)$ linear system with unknown $z = [x, y]^T$ of the form

$$Mz = c, \qquad \text{with } c \in \mathbb{R}^{n+1}.$$

(b) Assuming that A^{-1} is available, find a necessary and sufficient condition for $\widehat{A}$ to be invertible. (*Hint: by substitution or Gaussian elimination on the last row of M.*)

(c) Assume that A is such that we can solve a system of the form $Ax = b$ in $O(n)$ floating point operations (flops). What is the computational cost to solve a system $\widehat{A}x = \widehat{b}$ using Sherman-Morrison? Indicate also the flops on each step.

14. On the interval $\Omega = (0, 1)$, consider the two-point boundary value problem

$$\begin{aligned} &-au'' + bu' = f \qquad \text{in } \Omega, \\ &u(0) = u_0, u'(1) = 0, \end{aligned} \tag{9}$$

where the coefficients $a = a(x)$ and $b = b(x)$ are smooth functions satisfying $a(x) > 0$ and $b(x) \geq 0$ in $\overline{\Omega}$, and where the real forcing function $f = f(x)$ and boundary value u_0 are given.

(a) Consider uniform partitions (grids) of Ω in N sub-intervals $[x_{i-1}, x_i)$, $i = 1, \ldots, N$, of length $h = 1/N$. Derive the second order central finite difference discretisation operator $\mathcal{A}_h$, for (9) and write the corresponding finite difference method for the discrete solution vector $U = \{U_i\}_{i=0}^N$. Show that, indeed, the truncation error for this method is of $\mathcal{O}(h^2)$ if the exact solution u is regular enough.

(b) Let $a_i = a(x_i)$ and $b_i = b(x_i)$, $i = 0, 1, \ldots, N$. Consider for a moment the pure Dirichlet problem obtained by replacing the right-boundary condition with $u(1) = u_1$. Prove that, if h is small enough that $a_i \pm \frac{1}{2}hb_i \geq 0$ and that $\mathcal{A}_hU_i \leq 0$ ($\mathcal{A}_hU_i \geq 0$), $i = 1, \ldots, N$, then

$$\max_i U_i = \max\{U_0, U_N\} \qquad (\min_i U_i = \min\{U_0, U_N\}).$$

Comment on the implications of the above Discrete Maximum Principle (DMP) result on the existence and uniqueness of the solution to the discrete scheme and on the stability of the scheme. Now, consider again the original problem with Neumann condition at the right boundary. What can you say about the DMP in this case?

(c) Assume that the coefficients a and b are constant (you may also consider $u_0 = 0$ for simplicity.) Derive the standard continuous finite element method for (9) based on piecewise linear elements on the same uniform grid introduced above. Show that a specific implementation of the method yields, at least at the internal nodes, the same equations obtained with the finite difference method. Discuss the functional setting and range of applicability of both methods in terms of the regularity of the forcing function f.

(d) Assuming, moreover, that $b = 0$, show that a specific implementation of the finite volume method applied to (9) yields once more the finite difference method above.

15. Consider a linear neural network $f_\theta : \mathbb{R}^n \to \mathbb{R}^m$ defined as the parametrized function $f_\theta(x) = A_\theta x + b_\theta$, where $\theta = \{A, b\}$ are the weights and biases parameters, respectively given by $A_\theta \in \mathbb{R}^{m\times n}, b_\theta \in \mathbb{R}^m$, where $m \geq n$.

(a) By choosing the parameter as $\theta = \bar{\theta}$ we assume we can define the network as $f_{\bar{\theta}}(x) = Jx$, i.e. $J = A_{\bar{\theta}}$ with full column rank, and $b_{\bar{\theta}} = 0$. Given $y \in \mathbb{R}^m$, we want to solve the least square problem with objective function l defined as

$$\min_{x\in\mathbb{R}^n} l(x) \quad \text{where} \quad l(x) = \frac{1}{2}\|f_{\bar{\theta}}(x) - y\|^2.$$

(i) Derive the *normal equations* of the problem.

(ii) Exploit the Singular Value Decomposition (SVD) of J, namely $J = USV^T$, to show that the optimum is given by $x^* = VS^{-1}U^T y$.

(b) Assume now we have the training data pairs $\{(x_k, y_k)\}_{k=1}^K$. We define the loss function as

$$l(\theta) = \frac{1}{2K}\sum_{k=1}^{K} \|f_\theta(x_k) - y_k\|^2.$$

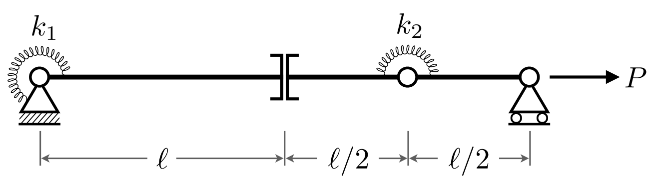
(iii) Compute the expression for the gradient of the loss function l_θ w.r.t. the weights and biases of the network, i.e. $\nabla_A l(\theta)$ and $\nabla_b l(\theta)$.

(iv) Write a pseudo-code algorithm for a gradient descent update of the parameter $\theta = \{A, b\}$.

Continuum Mechanics

16. Consider the plane and homogeneous deformation $y : [0,1] \times [0,1] \to \mathcal{E}$ such that, in Cartesian components, $y(0,0) = (0,0)$, $y(1,0) = (1,-3\alpha)$, and $y(0,1) = (3,\alpha)$, with α a scalar parameter. Determine the admissible values of the parameter α. For the case of $\alpha = 1$, determine the stretch and the rotation tensor in the right polar decomposition of the gradient of the deformation. Assuming linear elastic and isotropic material response, determine the state of stress for the admissible values of α and discuss the case of $\alpha = 1$.

17. Determine the buckling loads of the discrete elastic system shown in the figure below. Assume that the two torsional springs are linear elastic with stiffness k_1 and k_2. Note that the internal constraint allows only for the relative transverse motion between the connected bars.



18. An infinite, hyperelastic and incompressible strip of thickness h is adhered on a rigid flat substrate and subject under plane strain conditions to a reference shearing force of magnitude τ applied on its top surface. Take as reference strain energy density $\psi(\mathbf{F}) = \mu(\mathbf{F} \cdot \mathbf{F} - 3)/2$, where $\mathbf{F}$ is the deformation gradient and μ the shear modulus. Determine the deformation of the strip, the first Piola-Kirchhoff stress tensor, and the Cauchy stress tensor.

19. Two spheres of density ρ_s and radii r_1 and r_2 are connected at their center by a linear elastic spring of stiffness k and rest length ℓ, much greater than the spheres' radii. Assume that the two spheres are vertically aligned and steadily falling under the influence of gravity in a Newtonian fluid of viscosity μ and density ρ. Determine the velocity of the system and the elongation of the elastic spring. For the computation, assume Stokes flow in the surrounding fluid and neglect hydrodynamic interactions between the spheres. Comment about the cases of $r_1 = r_2$ and of $\rho_s = \rho$.

20. A straight elastic rod of length ℓ and circular cross section of radius r is in a state of pure bending through the application of moments at its ends. Let E denote the Young's modulus of the material of the rod and assume that this brakes as the relative rotation between its ends is of α_f. Determine the state of stress in the rod, its configuration at failure, and estimate the strength of the material, σ_f.