Ist. di Fisica Matematica mod. A
Second exercise session

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Exercises are numbered as in the lecture notes of the course.

1 Exercises from Chapter 2

Exercise 2.8.3. *Find the general solution to the following equations:*

$$x^2\, u_{xx} - y^2\, u_{yy} - 2y\, u_y = 0, \tag{1a}$$

$$x^2\, u_{xx} - 2xy\, u_{xy} + y^2\, u_{yy} + x\, u_x + y\, u_y = 0. \tag{1b}$$

Solution. To find the general solution to the two equations, it is convenient to put them in canonical form first. Let's consider equation (1a), and determine its characteristics. We have

$$a(x,y) = x^2, \quad b(x,y) = 0, \quad c(x,y) = -y^2$$

from which we obtain

$$b^2 - ac = x^2y^2 > 0,$$

hence the equation is of *hyperbolic* type. For simplicity, we assume to solve the equation in the domain

$$\Omega = \left\{ (x,y) \in \mathbb{R}^2 : x > 0, y > 0 \right\},$$

so that $\sqrt{b^2 - ac} = xy$ in Ω.

The characteristics are obtained by solving

$$\frac{dy}{dx} = \pm \frac{xy}{x^2} = \pm \frac{y}{x}$$

namely

$$u = \phi(x,y) = \frac{y}{x}, \quad v = \psi(x,y) = xy.$$

Inverting this relations we obtain

$$x = \sqrt{\frac{v}{u}}, \quad y = \sqrt{uv}.$$

Using the chain rule for differentiation yields to

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v} = -\frac{y}{x^2}\frac{\partial}{\partial u} + y\,\frac{\partial}{\partial v} = \\
&= -\sqrt{\frac{u^3}{v}}\,\frac{\partial}{\partial u} + \sqrt{uv}\,\frac{\partial}{\partial v}, \\
\frac{\partial}{\partial y} &= \frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v} = \frac{1}{x}\frac{\partial}{\partial u} + x\,\frac{\partial}{\partial v} = \\
&= \sqrt{\frac{u}{v}}\,\frac{\partial}{\partial u} + \sqrt{\frac{v}{u}}\,\frac{\partial}{\partial v}.
\end{aligned}$$

Notice that from this it follows that

$$-2y\,\frac{\partial}{\partial y} = -2u\,\frac{\partial}{\partial u} - 2v\,\frac{\partial}{\partial v}. \tag{2}$$

Using now Leibnitz rule and paying attention to the derivatives of the coefficients, we can compute the second derivative operators. We obtain

$$\begin{aligned}
\frac{\partial^2}{\partial x^2} &= \left(-\sqrt{\frac{u^3}{v}}\,\frac{\partial}{\partial u} + \sqrt{uv}\,\frac{\partial}{\partial v}\right)\left(-\sqrt{\frac{u^3}{v}}\,\frac{\partial}{\partial u} + \sqrt{uv}\,\frac{\partial}{\partial v}\right) = \\
&= \frac{u^3}{v}\,\frac{\partial^2}{\partial u^2} - 2u^2\,\frac{\partial^2}{\partial u\partial v} + uv\,\frac{\partial^2}{\partial v^2} + 2\frac{u^2}{v}\,\frac{\partial}{\partial u}, \\
\frac{\partial^2}{\partial y^2} &= \left(\sqrt{\frac{u}{v}}\,\frac{\partial}{\partial u} + \sqrt{\frac{v}{u}}\,\frac{\partial}{\partial v}\right)\left(\sqrt{\frac{u}{v}}\,\frac{\partial}{\partial u} + \sqrt{\frac{v}{u}}\,\frac{\partial}{\partial v}\right) = \\
&= \frac{u}{v}\,\frac{\partial^2}{\partial u^2} + 2\,\frac{\partial^2}{\partial u\partial v} + \frac{v}{u}\,\frac{\partial^2}{\partial v^2},
\end{aligned}$$

and consequently

$$\begin{aligned}
x^2\,\frac{\partial^2}{\partial x^2} &= u^2\,\frac{\partial^2}{\partial u^2} - 2uv\,\frac{\partial^2}{\partial u\partial v} + v^2\,\frac{\partial^2}{\partial v^2} + 2u\,\frac{\partial}{\partial u}, \\
-y^2\,\frac{\partial^2}{\partial y^2} &= -u^2\,\frac{\partial^2}{\partial u^2} - 2uv\,\frac{\partial^2}{\partial u\partial v} - v^2\,\frac{\partial^2}{\partial v^2}.
\end{aligned} \tag{3}$$

In conclusion, the canonical form of equation (1a) can be obtained by summing term by term (3) and (2), hence resulting (upon division by a common factor $-2v$) in

$$2u\,\varphi_{uv} + \varphi_v = 0. \tag{4}$$

To obtain the general solution to (1a), set

$$\Phi(u, v) := \varphi_v(u, v).$$

Equation (4), written in terms of Φ, becomes

$$2u\,\Phi_u + \Phi = 0$$

and hence can be easily solved, for example with the method of separation of variables. We obtain

$$\Phi(u,v) = \frac{k(v)}{\sqrt{u}}$$

and hence

$$\varphi(u,v) = f(u) + \frac{g(v)}{\sqrt{u}}$$

where g is a primitive of k. The general solution to (1a), written in terms of the initial variables (x,y), is then

$$\varphi(x,y) = f\left(\frac{y}{x}\right) + \sqrt{\frac{x}{y}}\,g(xy)$$

where f, g are two sufficiently regular (of class $\mathcal{C}^2$) arbitrary functions.

We now come to equation (1b). We have now

$$a(x,y) = x^2, \quad b(x,y) = -xy, \quad c(x,y) = y^2$$

from which we obtain

$$b^2 - ac = 0,$$

hence the equation is of *parabolic* type. For simplicity, we assume to solve the equation in the half-plane

$$\Omega = \left\{(x,y) \in \mathbb{R}^2 : x > 0\right\}.$$

The only characteristic can be obtained by solving

$$\frac{dy}{dx} = -\frac{xy}{x^2} = -\frac{y}{x}$$

namely

$$u = \phi(x,y) = xy.$$

For the second characteristic, we choose

$$v = \psi(x,y) = x.$$

Inverting these relations yields to

$$x = v, \quad y = \frac{u}{v}.$$

Using the chain rule, we obtain

$$
\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial u}{\partial x}\,\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\,\frac{\partial}{\partial v} = y\,\frac{\partial}{\partial u} + \frac{\partial}{\partial v} = \\
&= \frac{u}{v}\,\frac{\partial}{\partial u} + \frac{\partial}{\partial v}, \\
\frac{\partial}{\partial y} &= \frac{\partial u}{\partial y}\,\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\,\frac{\partial}{\partial v} = x\,\frac{\partial}{\partial u} = \\
&= v\,\frac{\partial}{\partial u}.
\end{aligned}
$$

Notice that from this it follows that

$$
\begin{aligned}
x\,\frac{\partial}{\partial x} &= u\,\frac{\partial}{\partial u} + v\,\frac{\partial}{\partial v}, \\
y\,\frac{\partial}{\partial y} &= u\,\frac{\partial}{\partial u}.
\end{aligned}
\tag{5}
$$

Using now Leibnitz rule and paying attention to the derivatives of the coefficients, we can compute the second derivative operators. We obtain

$$
\begin{aligned}
\frac{\partial^2}{\partial x^2} &= \left(\frac{u}{v}\,\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{u}{v}\,\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right) = \\
&= \frac{u^2}{v^2}\,\frac{\partial^2}{\partial u^2} + 2\,\frac{u}{v}\,\frac{\partial^2}{\partial u \partial v} + \frac{\partial^2}{\partial v^2}, \\
\frac{\partial^2}{\partial x \partial y} &= \left(v\,\frac{\partial}{\partial u}\right)\left(\frac{u}{v}\,\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right) = \\
&= u\,\frac{\partial^2}{\partial u^2} + v\,\frac{\partial^2}{\partial u \partial v} + \frac{\partial}{\partial u}, \\
\frac{\partial^2}{\partial y^2} &= \left(v\,\frac{\partial}{\partial u}\right)\left(v\,\frac{\partial}{\partial u}\right) = \\
&= v^2\,\frac{\partial^2}{\partial u^2},
\end{aligned}
$$

and consequently

$$
\begin{aligned}
x^2\,\frac{\partial^2}{\partial x^2} &= u^2\,\frac{\partial^2}{\partial u^2} + 2uv\,\frac{\partial^2}{\partial u \partial v} + v^2\,\frac{\partial^2}{\partial v^2}, \\
-2xy\,\frac{\partial^2}{\partial x \partial y} &= -2u^2\,\frac{\partial^2}{\partial u^2} - 2uv\,\frac{\partial^2}{\partial u \partial v} - 2u\,\frac{\partial}{\partial u}, \\
y^2\,\frac{\partial^2}{\partial y^2} &= u^2\,\frac{\partial^2}{\partial u^2}
\end{aligned}
\tag{6}
$$

In conclusion, the canonical form of equation (1b) can be obtained by summing term by term (6) and (5), hence resulting (upon division by a common factor v) in

$$
v\,\varphi_{vv} + \varphi_v = 0. \tag{7}
$$

To obtain the general solution to (1b), set

$$\Phi(u,v) := \varphi_v(u,v).$$

Equation (7), written in terms of Φ, becomes

$$v\,\Phi_v + \Phi = 0$$

and hence can be easily solved, for example with the method of separation of variables. We obtain

$$\Phi(u,v) = \frac{g(u)}{v}$$

and hence

$$\varphi(u,v) = f(u) + g(u)\,\ln(v).$$

The general solution to (1b), written in terms of the initial variables (x,y), is then

$$\varphi(x,y) = f(xy) + g(xy)\,\ln(x)$$

where f, g are two sufficiently regular (of class $\mathcal{C}^2$) arbitrary functions. $\diamondsuit$

2 Exercises from Chapter 3

Exercise 3.8.9 (The Gibbs phenomenon). *Denote*

$$S_n(x) = \frac{4}{\pi}\sum_{k=1}^{n}\frac{\sin((2k-1)x)}{2k-1}$$

the n-th partial sum of the Fourier series (3.5.35). Prove that

1. *for any $x \in (-\pi,\pi)$*

   $$\lim_{n\to\infty} S_n(x) = \operatorname{sign} x.$$

   Hint: derive the following expression for the derivative:

   $$S_n'(x) = \frac{2}{\pi}\frac{\sin(2nx)}{\sin(x)}.$$

2. *Verify that the n-th partial sum has a maximum at*

   $$x_n = \frac{\pi}{2n}.$$

3. *Prove that*

   $$S_n(x_n) = \frac{2}{\pi}\sum_{k=1}^{n}\frac{\pi}{n}\cdot\frac{\sin\left(\frac{(2k-1)\pi}{2n}\right)}{\frac{(2k-1)\pi}{2n}} \longrightarrow \frac{2}{\pi}\int_0^{\pi}\frac{\sin(x)}{x}\,dx \approx 1.17898$$

   for $n \to \infty$.

Thus for the trigonometric series (3.5.35)

$$\limsup_{n\to\infty} S_n(x) > 1 \quad \textit{for} \quad x > 0.$$

In a similar way one can prove that

$$\liminf_{n\to\infty} S_n(x) < -1 \quad \textit{for} \quad x < 0.$$

Solution. 1. Let's prove the hinted form for the derivative $S_n'(x)$. One has

$$S_n'(x) = \frac{4}{\pi}\sum_{k=1}^{n}\cos((2k-1)x)$$

hence we need to prove that

$$2\sin(x)\left(\cos(x)+\cos(3x)+\cdots+\cos((2n-1)x)\right) = \sin(2nx) \quad \text{for all } n \in \mathbb{N}.$$

The statement is true for $n = 1$, as $2\sin(x)\cos(x) = \sin(2x)$. We then proceed by induction, computing

$$\begin{aligned} 2\sin(x)\left(\cos(x)+\cos(3x)+\cdots+\cos((2n-1)x)+\cos((2n+1)x)\right) &= \\ = \sin(2nx) + 2\sin(x)\cos((2n+1)x) &= \\ = \sin(2nx) + 2\sin(x)\left(\cos(2nx)\cos(x) - \sin(2nx)\sin(x)\right) &= \\ = \sin(2nx)\left(1-2\sin^2(x)\right) + \cos(2nx)\sin(2x) &= \\ = \sin(2nx)\cos(2x) + \cos(2nx)\sin(2x) = \sin(2(n+1)x). \end{aligned}$$

We now have that

$$\begin{aligned} \frac{\sin(2nx)}{\sin(x)} &= \frac{\sin(2(n-1)x)\cos(2x)+\cos(2(n-1)x)\sin(2x)}{\sin(x)} = \\ &= \cos(2x)\frac{\sin(2(n-1)x)}{\sin(x)} + 2\cos(2(n-1)x)\cos(x), \end{aligned}$$

and consequently

$$S_n'(x) = \cos(2x)S_{n-1}'(x) + \frac{4}{\pi}\cos(2(n-1)x)\cos(x).$$

Fix $x > 0$. Since $S_n(0) = 0$ for all $n \in \mathbb{N}$, integrating the above relation on $[0, x]$ yields to

$$\begin{aligned}
S_n(x) &= \int_0^x \cos(2t)S'_{n-1}(t)\,dt + \frac{4}{\pi}\int_0^x \cos(2(n-1)t)\cos(t)\,dt = \\
&= [\cos(2t)S_{n-1}(t)]_{t=0}^{t=x} + 2\int_0^x \sin(2t)S_{n-1}(t)\,dt + \\
&\quad + \frac{4}{\pi}\frac{1}{2(n-1)}\int_0^{x2(n-1)} \cos(y)\cos\left(\frac{y}{2(n-1)}\right)dy = \\
&= \cos(2x)S_{n-1}(x) + 2\int_0^x \sin(2t)S_{n-1}(t)\,dt + \\
&\quad + \frac{4}{\pi}\frac{1}{2(n-1)}\int_0^{x2(n-1)} \cos(y)\cos\left(\frac{y}{2(n-1)}\right)dy.
\end{aligned}$$

Letting $n \to \infty$, we obtain that

$$S(x) := \lim_{n\to\infty} S_n(x) = \cos(2x)S(x) + 2\int_0^x \sin(2t)S(t)\,dt.$$

Upon differentiation of the above equality with respect to x, we get

$$S'(x)(1-\cos(2x)) + S(x)2\sin(2x) = 2\sin(2x)S(x) \iff S'(x)(1-\cos(2x)) = 0.$$

Since the above equation should be satisfied for all $x > 0$, we obtain that $S'(x) \equiv 0$, or equivalently $S(x)$ is constant. By evaluating

$$S(\pi/2) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2k-1} = 1$$

we obtain $S(x) = 1$ for all $x > 0$. Since S_n is an odd function of S, also S is odd, so we conclude that $S(x) = \operatorname{sign} x$.

2. To verify that $x_n = \pi/2n$ is a maximum for S_n, it suffices to show that

$$S'_n(x_n) = 0, \quad S''_n(x_n) < 0.$$

We have

$$S'_n(x_n) = \frac{2}{\pi}\frac{\sin(\pi)}{\sin(\pi/2n)} = 0$$

while

$$S''_n(x_n) = \frac{2}{\pi}\left.\frac{2n\,\cos(2nx)\sin(x) - \sin(2nx)\cos(x)}{\sin^2(x)}\right|_{x=\pi/2n} = \frac{2}{\pi}\frac{-2n}{\sin(\pi/2n)} < 0.$$

3. Now notice that

$$S_n(x_n) = \frac{2}{\pi}\sum_{k=1}^{n}\frac{\pi}{n}\cdot\frac{\sin\left(\frac{(2k-1)\pi}{2n}\right)}{\frac{(2k-1)\pi}{2n}}$$

is a Riemann sum for the continuous function $\frac{2}{\pi}\frac{\sin(x)}{x}$ on the interval $(0, \pi)$, with partition $\left\{0, \frac{\pi}{2n}, \frac{3\pi}{2n}, \dots, \frac{(2n-1)\pi}{2n}, \pi\right\}$. By definition of Riemann integral, this sum converges to

$$\frac{2}{\pi}\int_0^\pi \frac{\sin(x)}{x}\,dx$$

as wanted. $\diamondsuit$

Exercise 3.8.3 bis (A variation on Exercise 3.8.3). *For few instants of time $t \geq 0$ make a graph of the solution $u(x,t)$ to the wave equation on the half line $x \geq 0$ with* **Dirichlet** *boundary condition*

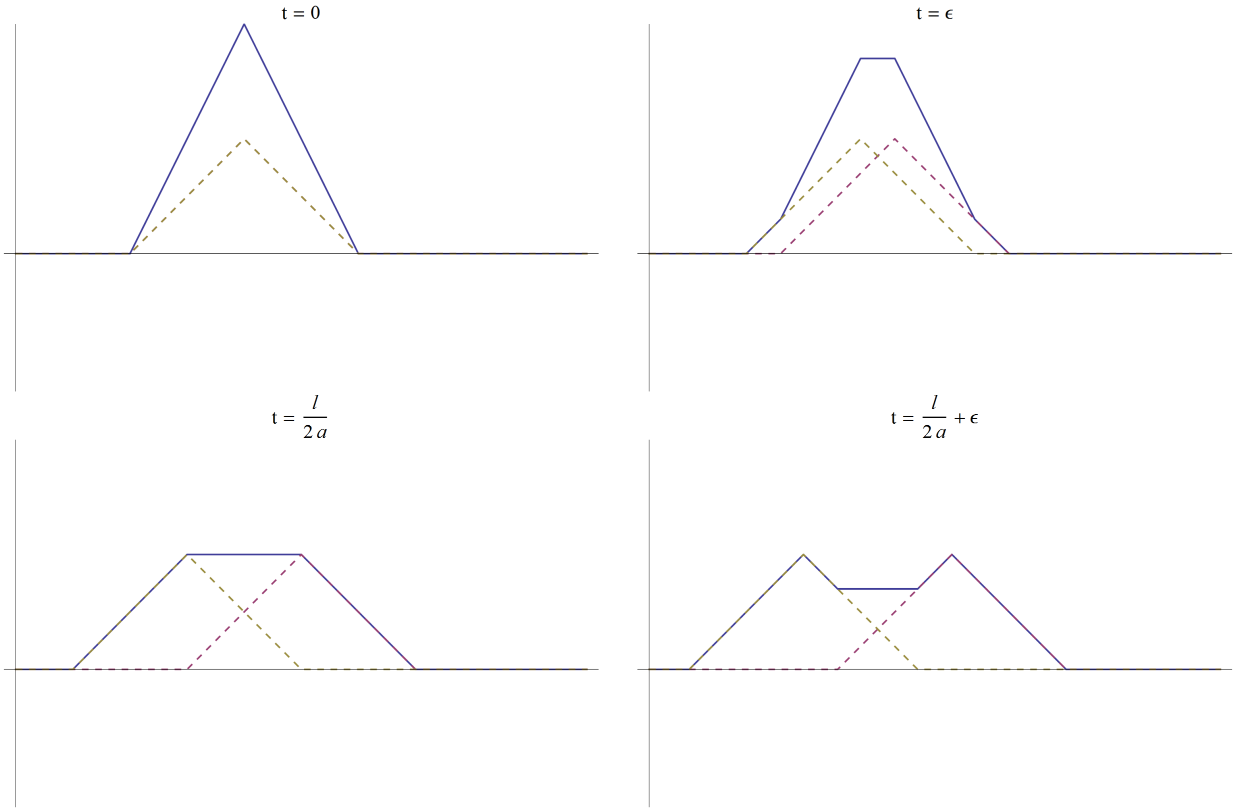
$$u(0,t) = 0$$

and with the initial data

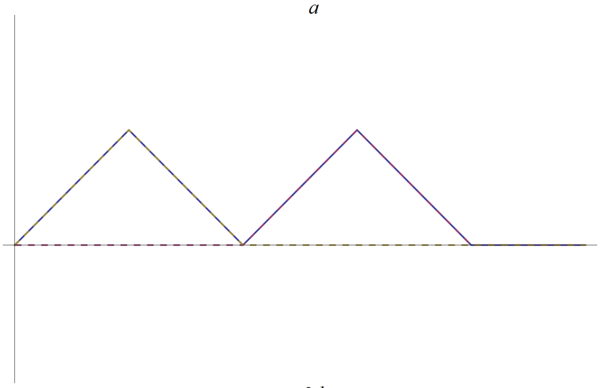
$$u(x,0) = \phi(x), \qquad u_t(x,0) = 0, \qquad x > 0$$

where the graph of the function $\phi(x)$ is an isosceles triangle of height 1 and base $[l, 3l]$.

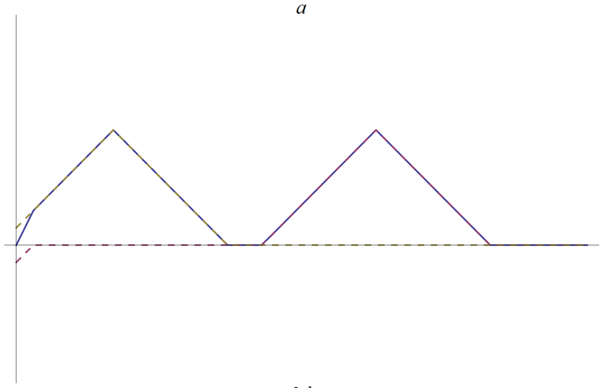
Solution. The analytic form of the solution can be computed from D'Alembert formula, extending the initial data on the half line $\{x < 0\}$ as an odd function. We plot here the graph of the solution with a continuous blue line. The yellow and purple dotted lines represent respectively the retarded and advanced wave, with halved height. $\diamondsuit$



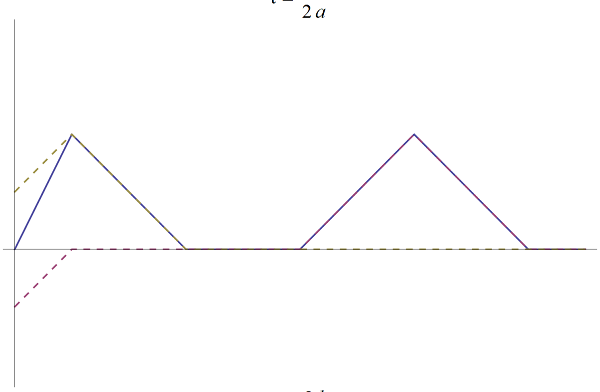
$t = \frac{l}{a}$



$t = \frac{l}{a} + \epsilon$



$t = \frac{3l}{2a}$



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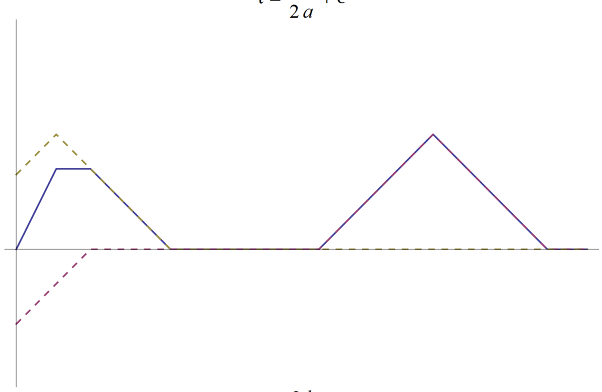

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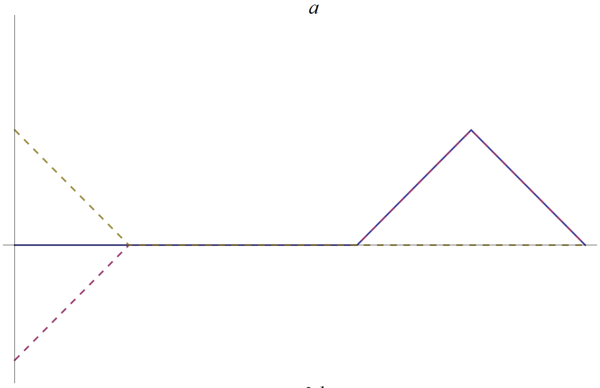
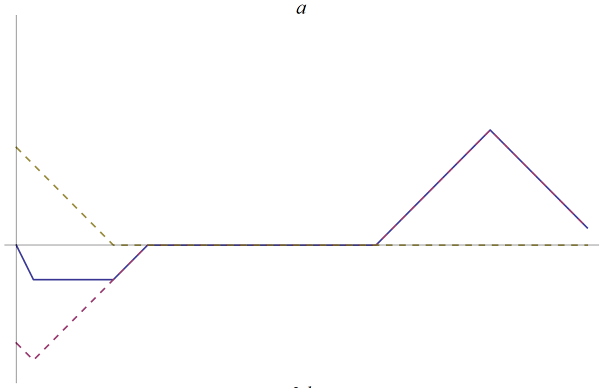
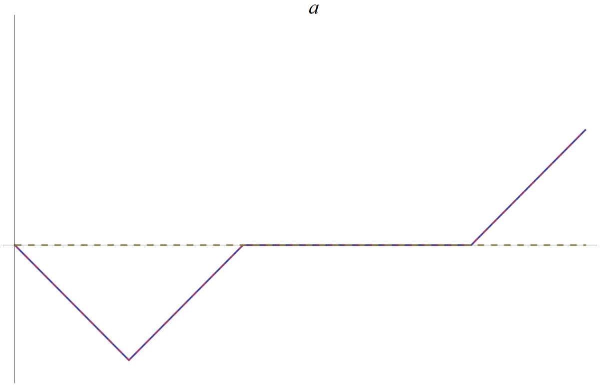


$t = \frac{2l}{a} + \epsilon$



$t = \frac{3l}{a}$



$t = \frac{3l}{a} + \epsilon$