NONCOMMUTATIVE GEOMETRY I

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When passing from classical to noncommutative spaces, we are forced to rethink about most of the classical concepts, which can be roughly arranged into three levels: measure theory, then topology and the most sophisticated one, geometry. The course supplies some basic functional analytic backgrounds for the first two levels. There are 5 lectures, that is 10 hours, in total. In the first three lectures (6 hours), we will give a quick introduction to Banach spaces, Banach algebras and then C^*-algebras [4, Ch. 1, 2, 4]. The highlight is the duality between locally compact Hausdorff spaces X and associated algebras of continuous coordinate functions $C(X)$ in Table 1.

Table 1. Gelfand-Naimark Duality

Topology	*Algebra*
locally compact Hausdorff space	C^*-algebra
compact space	unital C^*-algebra
compactification	unitalization
continuous proper map	$*$-homomorphism
homeomorphism	automorphism
open subset	ideal
closed subset	quotient algebra
metrizable	separable
Baire measure	positive linear functional

The remaining two lectures are devoted to compact operators [4, Ch. 5]. The common theme in both parts is the spectral theory and the corresponding functional calculus, which constitute fundamentals of the new calculus in which coordinate functions are quantized to operators (on Hilbert spaces). We conclude syllabus with another dictionary in Table 2 served as both motivations and guidelines of the course.
I choose [4] to be the main reference because of its condense style, supported by two other classical textbook on functional analysis [6] and [3]. Table 2 is taken from [5, 7]. Table 2 can be found in [2, §8] and [1, Page 20] with detailed comments on each entry.

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TABLE 2. Calculus and Infinitesimals

Classical	*Quantum*
complex variable	operator in a Hilbert space $\mathcal{H}$
real variable	self-adjoint operator
infinitesimal	compact operator
infinitesimal of order α	compact operator with characteristic values μ_n satisfying $\mu_n = O(n^{-\alpha})$ as $n \to \infty$
differential of real or complex variable	$\mathrm{d}f = [F, f] = Ff - fF$, where $(F, \mathcal{H})$ is a Fredholm module
integral of an infinitesimal of order 1	$⨍ T$ = coefficient of logarithmic divergence in the trace of T

References

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[5] José M. Gracia-Bondía, Joseph C. Várilly, and Héctor Figueroa. *Elements of noncommutative geometry.* Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Boston, Inc., Boston, MA, 2001.

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