2. Basic Representation Theory

Let G be a l.c (locally compact) group. We fix a Haar measure dx on G with the modular function $\Delta(x)$. As a result, L^p-functions, denoted by $L^p(G)$ or simply L^p, make sense. Equipped with the convolution and the involution:

$$f*g(x)=\int_G f(y)g(y^{-1}x)dy,\ f^*(x)=\Delta(x^{-1})\overline{f(x^{-1})}, \tag{2.1}$$

$L^1(G)$ becomes a Banach $*$-algebra. Any unitary representation π of G yields a $*$-representations of $L^1(G)$:

$$f\in L^1(G)\to\pi(f)=\int_G f(x)\pi(x)dx\in B(\mathcal{H}_\pi).$$

The convolution and involution are constructed in such a way that

$$\pi(f*g)=\pi(f)\pi(g),\ \pi(f^*)=\pi(f)^*,$$

and $\forall x\in G$:

$$\pi(x)\pi(f)=\pi(L_x f),\ \ \pi(f)\pi(x)=\Delta(x^{-1})\pi(R_{x^{-1}}f).$$

2.1. An important notion is *functions of positive type* on G. It consists of all $\varphi\in L^\infty(G)$ such that $\varphi(x)dx$ is a positive linear functional on $L^1(G)$:

$$\int(f^**f)\varphi\,dx\geq 0,\ \ \forall f\in L^1(G).$$

Show that the condition above means

$$\iint f(x)\overline{f(y)}\varphi(y^{-1}x)dy\,dx\geq 0,\ \ \forall f\in L^1(G).$$

2.2. Denote by $\mathcal{P}(G)$ the collection of all *continuous functions of positive type.* For any unitary representation π of G, and $u\in\mathcal{H}_\pi$. Show that the function $x\in G\to\varphi(x)$ belongs to $\mathcal{P}(G)$ with $\varphi(x)=\langle\pi(x)u,u\rangle$.

2.3. Let $f\in L^2(G)$ and $\tilde f(x)=\overline{f(x^{-1})}$. Show that $f*\tilde f\in\mathcal{P}(G)$.
Hint: consider the left regular representation.

When G is discrete, the delta-distribution at the $e\in G$ is the unit element of the algebra $L^1(G)$. Otherwise, $L^1(G)$ has only "approximate identity".

2.4. Let $\mathcal{U}$ be the collection of open neighborhoods of the unit $e\in G$. For each $U\in\mathcal{U}$, let ψ_U be a function on G such that:

i) $\operatorname{supp}\psi_U\subset U$ and is compact;
ii) $\int\psi_U=1$ and $\psi_U\geq 0$.

Show that, as $U\to\{e\}$,

$$\left\|\psi_U*f-f\right\|_p\to 0,\ 1\leq p<\infty$$

If in addition $\psi_U(x) = \psi_U(x^{-1})$, $\forall x \in G$, show that

$$\left\| f * \psi_u - f \right\|_p \to 0,\ 1 \le p < \infty,$$

as $U \to \{e\}$.
Hint: One might need Minkowski's inequality for integrals.

Let us examine that any non-zero function of positive type arises from a unitary representation. Given $\varphi \neq 0$ of positive type, it defines a positive semi-definite Hermitian form on $L^1(G)$:

$$\left\langle f, g \right\rangle_\varphi = \int (g^* * f)\varphi = \iint f(x)\overline{g(y)}\varphi(y^{-1}x)\,dx\,dy.$$

Let $\mathcal{N} = \{f \in L^1(G) : \left\langle f, f \right\rangle_\varphi = 0\}$ be the null space. Note that $f \in \mathcal{N}$ implies $\left\langle f, g \right\rangle_\varphi = 0$ for all $g \in L^1(G)$ (by Schwartz inequality). The form induces a inner product on the quotient space $L^1/\mathcal{N}$ and the completion yields a Hilbert space H_φ.

2.5. For $f \in L^1(G)$, let $[f] \in L^1/\mathcal{N}$ be the quotient class. Show that
1) $\left\|[f]\right\|_{H_\varphi} \le \left\|\varphi\right\|_\infty \left\|f\right\|_{1/2}$;
2) for $f, g \in L^1(G)$, $x \in G$, we have $\left\langle L_x f, L_x g \right\rangle_\varphi = \left\langle f, g \right\rangle_\varphi$.

It follows that the left translation L_x gives rise to a unitary representation on H_φ and the corresponding $*$-representation of $L^1(G)$ is given by $\pi_\varphi(f)([g]) = [f * g]$.

2.6. Keep the notations as above. Let $\{\psi_U\}$ be an approximate identity in $L^1(G)$ (cf. Ex. 2.4).

(1) Show that $[\psi_U]$ converges weakly in H_φ to an element ϵ such that $\left\langle [f], \epsilon \right\rangle_\varphi = \int f\varphi$ for all $f \in L^1$.
Hint: start with

$$\left\langle [f], [\psi_U] \right\rangle_\varphi = \int (\psi_U^* * f)\varphi,\ f \in L^1.$$

(2) Show that $\forall f, g \in L^1$:

$$\left\langle [f], [g] \right\rangle_\varphi = \left\langle [g], \pi_\varphi(f)\epsilon \right\rangle_\varphi.$$

(3) Show that ϵ is a cyclic vector, that is, the linear span of $\{\pi_\varphi(x)\epsilon : x \in G\}$ is dense in H_φ.
(4) Show that

$$\left\langle \pi_\varphi(x)\epsilon, \epsilon \right\rangle = \varphi(x) \text{ locally a.e. (almost everywhere)}. \tag{2.2}$$

It follows that every function of positive type agrees locally a.e. with a continuous function.
Hint: Eq. (2.2) means

$$\int \left\langle \epsilon, \pi_\varphi(\cdot)\epsilon \right\rangle \bar{f} = \int \bar{\varphi}\bar{f},\ \forall f \in L^1.$$

(5) Assume $\psi \in \mathcal{P}(G)$, show that $\left\|\psi\right\|_\infty = \psi(1)$ and $\psi(x^{-1}) = \overline{\psi(x)}$.

We point out that $\mathcal{P}$ is a convex cone as well as $\mathcal{P}_1$:

$$\mathcal{P}_1 = \{\varphi \in \mathcal{P} : \varphi(1) = 1\}.$$

2.7. Assume that $\varphi \in \mathcal{P}_1$ and the representation (π_φ, H_φ) discussed before is reducible: $H_\varphi = M \oplus M^\perp$, where both factors are nontrivial G-space.

(1) Show that the cyclic vector ϵ can be written as $\epsilon = \alpha u + \beta v$, with $u \in M$ and $v \in M^\perp$ are unit vectors and $\alpha, \beta > 0$.

(2) Let $\varphi_1(x) = \left\langle \pi_\varphi(x)u, u \right\rangle_\varphi$, $\varphi_2(x) = \left\langle \pi_\varphi(x)v, v \right\rangle_\varphi$. Show that φ_1 and φ_2 are linear independent. Moreover, $\varphi = \alpha^2 \varphi_1^2 + \beta^2 \varphi_2^2$.

2.8. Assume that $\varphi \in \mathcal{P}_1$ and the representation (π_φ, H_φ) discussed before is irreducible and $\varphi = \varphi_1 + \varphi_2$ with $\varphi_1, \varphi_2 \in \mathcal{P}_1$. Show that φ_1 and φ_2 are linear dependent.

2.9. Show that

(1) The linear span of $C_c(G) \cap \mathcal{P}(G)$ contains of all functions of the form $f * g$, $f, g \in C_c(G)$ ($C_c(G)$ denotes functions with compact support).
Hint: we have seen $f * \tilde{f} \in \mathcal{P}(G)$, where $\tilde{f}(x) = \overline{f(x^{-1})}$.

(2) The linear span is dense in $C_c(G)$ in the uniform norm and in $L^p(G)$ in the L^p-norm, $1 \le p < \infty$.