1. One Parameter Groups of Operators

One parameter groups of operators $\{V_t\}$, $t \in R$ are known as strongly continuous representations of the Lie group $\mathbb{R}$ on some Banach space B. To be precise, for each t, $V_t : B \to B$ is a bounded operator, such that

$$V_s V_t = V_{s+t}, \ s, t \in \mathbb{R} \tag{1.1}$$

$$V_0 = 1. \tag{1.2}$$

The strong continuity means that $t_n \to t$ implies $V_{t_n}(u) \to V_t(u)$ in B for each $u \in B$.

1. Let $T_n : B_1 \to B_2$, $n = 0, 1, 2, \ldots$, be a sequence of uniformly bounded set of linear operators between Banach spaces. Let $L \subset B_1$ be a dense linear subspace. Suppose for each $u \in L$, $T_n u \to T_0 u$ as $n \to \infty$ in the B_2-norm. Show that the convergence holds for all $u \in B_1$.

2. Show that the translation group acting on $L^p(\mathbb{R})$, $1 \le p < \infty$:

$$T_s(f)(x) = f(x - s), \ s, t \in \mathbb{R} \tag{1.3}$$

is an example of one parameter group of operators.
Hint: To show the strong continuity, check the convergence for $f \in C_0^\infty(\mathbb{R})$ (smooth functions vanishing at ∞) and then apply the previous exercise.

A one parameter group V_t, $t \in \mathbb{R}$, on B has associated an infinitesimal generator A, often an unbounded operator on B, defined by

$$Au = \lim_{h \to 0} \frac{V_h(u) - u}{h}. \tag{1.4}$$

"Unbounded" means A is not well-defined for all $u \in B$, the domain of A:

$$\mathcal{D}(A) = \{u \in B : \text{the limit in Eq. (1.4) exists in } B\}. \tag{1.5}$$

We write, symbolically, $V_t = e^{tA}$.

3. Show that $V_t(\mathcal{D}(A)) \subset \mathcal{D}(A)$ and

$$AV_t(u) = \frac{dV_t}{dt} V_t(u), \ \forall u \in \mathcal{D}(A).$$

4. Show that A is a densely defined, closed operator. Namely:

(1) $\mathcal{D}(A)$ is dense in B; Hint: $\forall u \in B$, consider

$$u_\varepsilon = \varepsilon^{-1} \int_0^\varepsilon V_t(u) dt$$

as an approximation.

(2) The graph

$$\mathcal{G}_A = \{(u, Au) \in B \oplus B : u \in \mathcal{D}(A)\}$$

is closed in $B \oplus B$. In other words,

$$u_n \in \mathcal{D}(A), \ u_n \to u, \ A(u_n) \to v \text{ in } B$$

implies $u \in \mathcal{D}(A)$ and $Au = v$.

5. For $\rho(t)$ a function on $\mathbb{R}$, we denote, symbolically, the operator $\hat{\rho}(-iA)$

$$\hat{\rho}(-iA)(u) = \int_{\mathbb{R}} \rho(t) V_t(u) dt, \ u \in B.$$

(1) For $\rho \in C_0^\infty(\mathbb{R})$, show that for $k \in \mathbb{N}$

$$A^k \hat{\rho}(-iA)(u) = \int_{\mathbb{R}} \rho^{(k)}(t) V_t(u) dt.$$

In particular, $\hat{\rho}(-iA)(u)$ is a C^∞-vector belonging to the domain of A^k for all k.

(2) Choose $\rho \in C_0^\infty(\mathbb{R})$ such that $\int \rho(t) dt = 1$, set $\rho_j(t) := j\rho(jt)$, thus, $\rho_j \in C_0^\infty(\mathbb{R})$. Show that

$$\widehat{\rho_j}(-iA)(u) = \int_{\mathbb{R}} \rho_j(t) V_t(u) dt \to u, \ \text{as } j \to \infty.$$

6. Let V_t, $t \in R$ be a one-parameter group of operators on B. Show

(1) the locally uniform boundedness on the norm:

$$\|V_t\| \leq M, \ \forall |t| \leq 1$$

for some $M \in [1, \infty)$. Hint: uniform boundedness theorem.

(2) furthermore, we have

$$\|V_t\| \leq M e^{K|t|}$$

for some K, say $K = \log M$.

(3) let $\lambda \in \mathbb{C}$ with $\Re\lambda > K$, then λ belongs to the resolvent set of A, that is $(A - \lambda)^{-1}$ exists. Moreover, we have

$$(\lambda - A)^{-1}(u) = R_\lambda(u) := \int_0^\infty e^{-\lambda t} V_t(u) dt, \ u \in B.$$

Hint: Show $R_\lambda(\lambda - A)$ is the identity on $\mathcal{D}(A)$ and $(\lambda - A)R_\lambda$ is the identity on B.

(4) the existence of the resolvent implies the closedness of A.

The uniqueness of the infinitesimal generator.

7. If V_t and W_t are two one-parameter group on some Banach space B with the same infinitesimal generator, then $V_t = W_t$ for all $t \in \mathbb{R}$.
Hint: fix $u \in B$ and $w \in B^*$ (the dual space), consider functions $f_V(t) = (V_t(u), w)$, $f_W(t) = (W_t(u), w)$, show their Laplace transforms are equal. Finally, make use of Hahn-Banach theorem.