3. Harmonic Analysis on Locally Compact Abelian Groups

Let G be a l.c.a (locally compact abelian) group, primary examples are $G = \mathbb{R}, \mathbb{Z}, \mathbb{T}, \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. By Schur's lemma, all the irreducible representations of G are one dimensional. They are parametrized by *characters* of G, which are continuous group homomorphisms from G to the unit circle $\mathbb{T}$. Denote by $\widehat{G}$ the space of characters which is also belong to the category of l.c.a groups with respect to the pointwise multiplication and compact-open topology.

We fix a Haar measure dx on G and denote by $L^1(G)$ which is a Banach $*$-algebra regarding to the convolution product $*$. We can identify $\xi \in \widehat{G}$ as a functional on $L^1(G)$: $f \to \bar{\xi}(f) = \xi^{-1}(f) = \int_G \overline{\langle x, \xi \rangle} f(x) dx$, where $\langle x, \xi \rangle = \xi(x)$ is the evaluation. The Fourier transform becomes a special case of the Gelfand transform: $L^1(G) \to C(G)$ defined by:

$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) = \int_G \overline{\langle x, \xi \rangle} f(x) dx.$$

On the other hand, every multiplicative linear functional comes from integration against a character.

3.1. Due to the duality between L^1 and L^∞, any continuous linear functional $\Phi \in (L^1(G))^*$ is given by integration against some $\phi \in L^\infty(G)$. Show that

1. For any $f \in L^1(G)$, $x \in G$, $\Phi(f)\phi(x) = \Phi(L_x f)$ locally a.e.. In particular, when $\Phi(f) \neq 0$, the function (in x): $\Phi(L_x f)/\Phi(f)$ is a continuous representative of ϕ. (Hint: the multiplicative property of Φ says: $\Phi(f)\Phi(g) = \Phi(f * g)$.)
2. $|\phi(x)| = 1$ and ϕ is multiplicative: $\psi(xy) = \phi(x)\phi(y)$, $\forall x, y \in G$.

In conclusion, $\widehat{G}$ is the spectrum of $L^1(G)$ in the Gelfand theory and we have a $*$-algebra homomorphism $\mathcal{F} : L^1(G) \to C_0(\widehat{G})$ (continuous functions vanishing at infinity) with dense range.

A few more words about the topology of $\widehat{G}$. It can be shown that the compact open topology agrees with the weak $*$-topology inherited from $L^\infty(G)$. The weak $*$-convergence for functionals $\Phi_n \to \Phi$ means $\Phi_n(f) \to \Phi(f)$, for all $f \in L^1(G)$. A nice property of the weak $*$-topology is the Banach-Alaolu's theorem: the closed unit ball is compact. It follows that $\widehat{G}$ is locally compact since $\widehat{G} \cup \{0\}$ is a closed subset of the closed unit ball.

3.2. Assume G is compact and the Haar measure satisfies $|G| = 1$. Show that $\widehat{G}$ is an orthonormal set in $L^2(G)$

3.3. Show that if G is compact, then $\widehat{G}$ is discrete while G is discrete, then $\widehat{G}$ is compact. (Hint: if G is discrete, then $L^1(G)$ is unital, make use of the Gelfand theory; if G is compact, then $f \equiv 1 \in L^1$, try to show that $\{1\}$ is open in $\widehat{G}$ using $\{g \in L^\infty : \left|\int g\right| > 1/2\}$ is a weak $*$-open set. The orthonormal property proved in the previous exercise might be useful.)

Furthermore, $\mathcal{F}$ can be extended to $M(G)$, complex Radon measures on G and landed in $B(\widehat{G})$ (bounded continuous functions):

$$\mu(\xi) = \int_G \overline{\langle \xi, x \rangle} d\mu(x), \ \forall \mu \in M(G). \tag{3.1}$$

3.4. Show the well-known formula $\mathcal{F}(\mu * \nu) = (\mathcal{F}(\mu))(\mathcal{F}(\nu))$.
Recall the convolution between measures: $\forall\, \nu, \mu \in M(G)$, $\psi \in C_0(G)$:

$$\int \psi d(\mu * \nu) = \iint \psi(xy) d\mu(x) d\nu(y).$$

With the convolution, $M(G)$ as the dual space (continuous linear functionals) of the Banach space $C_0(G)$, becomes a Banach $*$-algebra. The Fourier transform Eq. (3.1) makes $\widehat{G}$ into a part of the spectrum of $M(G)$, that is $\mathcal{F}(\mu)$ is the restriction of the Gelfand map (from $M(G)$ to functions on its spectrum) onto $\widehat{G}$.

Consider a similar construction (the inverse Fourier transform) sending $\mathsf{m} \in M(\widehat{G})$ to a function φ_{m} on G:

$$\varphi_{\mathsf{m}}(x) = \int_{\widehat{G}} \langle x, \xi \rangle \, d\mathsf{m}(\xi).$$

3.5. We denote the map by $\mathsf{F} : M(\widehat{G}) \to \mathcal{B}(G)$: $\mathsf{F}(\mathsf{m}) = \varphi_{\mathsf{m}}$, where $\mathcal{B}(G)$ is the collection of all such functions. Show that

(1) $\mathcal{B}(G) \subset C_b(G)$, where $C_b(G)$ denotes all the bounded continuous functions and F is norm-decreasing with respect to the uniform norm of $C_b(G)$.
(2) F is injective.

3.6. If $\mathsf{m} \in M(\widehat{G})$ is a positive measure, show that φ_{m} is a function of positive type.

The converse is one of the fundamental results of the theory, known as Bochner's Theorem. Namely, for any $\varphi \in \mathcal{P}(G)$, there exits a unique positive measure $\mathsf{m} \in M(\widehat{G})$ such that $\varphi = \varphi_{\mathsf{m}}$. It follows that $\mathcal{B}(G)$ is the linear span of $\mathcal{P}(G)$. In particular, it includes functions of the form $f * g$, $f, g \in L^1(G)$, see the previous homework set Exercise 2.9. Set $\mathcal{B}^p(G) = \mathcal{B}(G) \cap L^p(G)$, $1 \le p < \infty$.

Let us consider the inverse of F: $\mathsf{F}^{-1}(f) = \mathsf{m}_f$ formally we expect

$$f(x) = (\mathsf{F}^{-1} \circ \mathsf{F})(f)(x) = \int_{\widehat{G}} \langle x, \xi \rangle \, d\mathsf{m}_f(\xi). \tag{3.2}$$

To reach the Fourier inverse theorem, one needs to construction a dual measure $d\xi$ on $\widehat{G}$ such that $d\mathsf{m}_f(\xi) = \mathcal{F}(f)(\xi) d\xi$ for all suitable f.

3.7. Let $h \in C_c(G)$ with $\mathcal{F}(h)(1) = \int h = 1$, put $g = h^* * h$.

(1) Show that $\mathcal{F}(g) \ge 0$ and $\mathcal{F}(g)(1) = 1$. In particular, there is a small neighborhood V of the unit of $\widehat{G}$ on which $\mathcal{F}(g) > 0$.
(2) Show that for any $K \subset \widehat{G}$ compact, there exits $f \in C_c(G) \cap \mathcal{P}(G)$ such that $\mathcal{F}(f) \ge 0$ on $\widehat{G}$ and $\mathcal{F}(f) > 0$ on K. (Hint: cover K by translations of V in the previous part: $K \subset \cup_1^n x_j V$ and consider $f = (\sum_1^n x_j) g$.)

3.8. Keep notations in Eq. (3.2). For $f, g \in \mathcal{B}^1 = \mathcal{B} \cap L^1$, show that $\mathcal{F}(f) d\mathsf{m}_f = \mathcal{F}(g) d\mathsf{m}_g$.

Here is the first version of the Fourier inversion formula.

3.9. Keep notations in Eq. (3.2). Consider the following linear functional $I : C_c(\widehat{G}) \to \mathbb{C}$. Given $\psi \in C_c(\widehat{G})$, according to the Exercise 3.7, we can find $f \in L^1(G) \cap \mathcal{P}(G)$ such that $\mathcal{F}(f) > 0$ on the support of ψ. We define

$$I(\psi) = \int_{\widehat{G}} \frac{\psi}{\mathcal{F}(f)} d\mathsf{m}_f. \tag{3.3}$$

(1) Show that I is well-defined, that is Eq. (3.3) is independent of the choice of f.
(2) Show that I is translation invariant: $I(L_\eta(\psi)) = I(\psi)$ for any $\psi \in C_c(\widehat{G})$ and $\eta \in \widehat{G}$.
(3) The functional $I(\psi) = \int \psi d\xi$ is the dual Haar measure $d\xi$ on $\widehat{G}$. Show that for any $f \in \mathcal{B}^1$, $\mathcal{F}(f) \in L^1$ and $\mathcal{F}(f)(\xi) d\xi = d\mathsf{m}_f$. In other words,

$$f(x) = \int_{\widehat{G}} \langle x, \xi \rangle \, \mathcal{F}(f)(\xi) d\xi.$$

The natural pairing $\langle x, \xi \rangle$ between $x \in G$ and $\xi \in \widehat{G}$ is reflexive, therefore we have a group homomorphism $\Psi : G \to \widehat{\widehat{G}}$ from G to its double dual. The Pontrjagin Duality asserts that Ψ is indeed an isomorphism of topological groups. The first application of the duality is the following improved version of the Fourier inversion formula.

3.10. Suppose $f \in L^1(G)$ and $\mathcal{F}(f) \in L^1(\widehat{G})$.

(1) Show that $\widehat{f} \in \mathcal{B}^1(\widehat{G})$ and $d\mathsf{m}_{\widehat{f}} = f(x^{-1}) dx$.
(2) Show that $f(x) = \mathcal{F}^2(f)(x^{-1})$ a.e. (almost everywhere), that is

$$f(x) = \int \langle x, \xi \rangle \, \mathcal{F}(f)(\xi) d\xi, \text{ for a.e. } x.$$