4. Homogeneous spaces and Poisson Summation Formula

Let G be a l.c. (locally compact) group and $H \subset G$ be a closed subgroup. We'd like to study the quotient space G/H starting with the existence of G-invariant measures. The answer is not always affirmative. For instance, take G to be the affine transformations on $\mathbb{R}$ consisting of dilations and translations and H to be the subgroups of dilations. The resulting G-homogeneous space is $\mathbb{R}$, on which the translation invariants measures are the Lebesgue measure upto scalar multiples. They are not invariant under dilations.

Let dg and dh be left-Haar measures on G and H respectively and Δ_G and Δ_H are the modular functions. Denote by $q : G \to G/H$ the quotient map: $q(g) = gH$. Integration along the fiber yields $P : C_c(G) \to C_c(G/H)$:

$$(Pf)(gH) = \int_H f(gh)dh.$$

Notice that the left-invariance of dh is required to make sure P is well-defined.

One can show that P is is surjective. More precisely, $\forall \in \phi \in C_c(G/H)$, there exists $f \in C_c(G)$ such that $P(f) = \phi$ and $q(\operatorname{supp} f) = \operatorname{supp} \phi$. Also, if $\phi \geq 0$, then $f \geq 0$.

4.1. Suppose there exists a left G-invariant measure μ on G/H. Show that

(1) Upto suitable normalization, the measure μ is completely determined by the equation:

$$\int_G f(g)dg = \int_{G/H} P(f)d\mu = \int_{G/H} \int_H f(gh)dhd\mu(gH). \tag{4.1}$$

(2) $\Delta_G|_H = \Delta_H$;

4.2. Keep notations as above. Suppose $\Delta_G|_H = \Delta_H$.

(1) Show that if $f \in C_c(G)$ and $P(f) = 0$, then $\int_G f(g)dg = 0$.
(2) Show that $\forall f_1, f_2 \in C_c(F)$, $P(f_1) = P(f_2)$ implies $\int_G f_1 = \int_G f_2$.
(3) Show that Eq. (4.1) leads to a left G-invariant Radon measure μ on G/H.

When the left G-invariant measure on G/H does not exist, a weaker replacement is quasi-invariant. For $g \in G$ denote by μ_g the left translation of the measure μ on G/H, that is $\mu_g(E) = \mu(gE)$ for any measurable set E. The quasi-invariance of μ means all left translations μ_g are equivalent, that is, mutually absolutely continuous. We say μ is strongly quasi-invariant if there is a continuous function $\lambda : G \times G/H \to (0, \infty)$ such that

$$d\mu_g(p) = \lambda(g, p)d\mu(p), \ \forall x \in G, \ p \in G/H. \tag{4.2}$$

In other words, the Radon-Nikodym derivative $(d\mu_g/d\mu)(p)$ is jointly continuous in g and p.

4.3. Suppose there exist a strongly quasi-invariant measure μ on G/H with Radon-Nikodym derivative $(d\mu_g/d\mu)(p) = \lambda(g, p)$.

(1) Show that for $g_1, g_2 \in G$, $\lambda(g_1g_2, p) = \lambda(g_1, g_2p)\lambda(g_2, p)$ for l.e. $p \in G/H$. (Hint: use $\mu_{g_1g_2} = (\mu_{g_1})_{g_2}$.)

(2) Show that

$$f \in C_c(G) \to \int_{G/H} \int_H f(gh)\lambda(gh,H)^{-1} dh d\mu(gH)$$

defines a left G-invariant positive linear functional on $C_c(G)$. It follows that, there is a $c > 0$ such that

$$\int_{G/H} \int_H f(gh)\lambda(gh,H)^{-1} dh d\mu(gH) = c \int_G f(g) dg. \tag{4.3}$$

(3) Put $\rho(g) = c\lambda(g,H)$ where c is the constant found in the previous step. Show that ρ is a continuous and positive function with the properties:

$$\int_G f(g)\rho(g)dg = \int_{G/H} \int_H f(gh) dh d\mu(gH),\ \forall f \in C_c(G) \tag{4.4}$$

$$\rho(gh) = \Delta_G(h)^{-1}\Delta_H(h)\rho(g),\quad g \in G, h \in H. \tag{4.5}$$

A continuous and positive function ρ satisfying Eq. (4.5) is called a rho-function on G. We shall work out the opposite direction that any rho-function gives rise to a strongly quasi-invariant measure μ on G/H, starting with the functional:

$$P(f) \in C_c(G/H) \to \int_G f(g)\rho(g)dg \tag{4.6}$$

gives rise to a strongly quasi-invariant measure μ on G/H. (Recall that $\{P(f) : f \in C_c(G)\}$ is a dense subset in $C_c(G/H)$.)

4.4. Given a rho-function ρ on G and denote by μ the measure given by the function in Eq. (4.6). Put $\lambda(g_1, g_2H) = \rho(g_1g_2)/\rho(g_2)$

(1) Show that $\lambda(g_1, g_2H)$ is well-defined and yields a continuous function $\lambda : G \times G/H \to (0,\infty)$.
(2) Show that λ is the Radon-Nikodym derivatives regarding to the left translations of μ, that is Eq. (4.2) holds.

From now on, we assume that G is a l.c.a (locally compact abelian) group and $H \subset G$ a closed subgroup.

4.5. Denote by:

$$H^\perp = \{\xi \in \widehat{G} : \langle h, \xi\rangle = 1 \text{ for all } h \in H\ \}.$$

Show that $(H^\perp)^\perp = H$. (Hint: To show $(H^\perp)^\perp \subset H$, one might need the fact (Gelfand-Raikov Theorem) that for l.c groups, irreducible unitary representations separate points. In our case, characters separate points.)

We point out two important isomorphisms. First, $(G/H)^\wedge \cong H^\perp$:

$$\Psi : (G/H)^\wedge \to H^\perp : \eta \to \eta \circ q$$

where q is the quotient map. If we replace G by $\widehat{G}$ and H by $H^\perp$, then $(\widehat{G}/H^\perp)^\wedge \cong (H^\perp)^\perp \cong H$. The explicit map $H \to (\widehat{G}/H^\perp)^\wedge$ is given by $x \to \eta_x$ where $\left\langle \eta_x, \xi H^\perp \right\rangle := \langle x, \xi \rangle$. Now the Pontrjagin duality yields the second isomorphism: $\widehat{G}/H^\perp \cong \widehat{(\widehat{G}/H^\perp)}\widehat{} \cong \widehat{H}$. More precisely:

$$\Phi : \widehat{G}/H^\perp \to \widehat{H} : \xi H^\perp \to \xi|_H.$$

4.6. We choose left-invariant measures dg, dh and μ so that Eq. (4.1) holds. For $f \in C_c(G)$, denote by $F = P(f)$ as before, that is $F(gH) = \int_H f(gh)dh$.

1) Show that $\widehat{F} = \widehat{f}|_{H^\perp}$ regarding to the identification $(G/H)^\wedge \cong H^\perp$.
2) If we further assume $\widehat{f}|_{H^\perp} \in L^1(H^\perp)$, show the Poisson summation formula:

$$\int_H f(gh)dh = \int_{H^\perp} \widehat{f}(\xi) \langle x, \xi \rangle \, d\xi, \tag{4.7}$$

where the Haar measures on H and $H^\perp$ have to be suitably normalized.
(Hint: make use of the Fourier inversion formula in the previous homework set.)

Two remarks:

i) If we start with $f \in L^1(G)$, then F is well-defined a.e. (almost every). The results of the exercise holds a.e..
ii) To recover the classical Poisson summation formula, take $G = \mathbb{R}$ and $H = \mathbb{Z}$, while $\widehat{\mathbb{R}} \cong \mathbb{R}$ via $\langle x, \xi \rangle = e^{2\pi i x \xi}$ so that $H^\perp \cong \mathbb{Z}$. In this case, Eq. (4.7) reads

$$\sum_{n \in \mathbb{Z}} f(x+n) = \sum_{n \in \mathbb{Z}} \widehat{f}(n) e^{2\pi i n x}, \ \forall x \in \mathbb{R}.$$