KP τ-functions and their applications

SISSA, April-May 2014

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About the course

The material is based on two courses given by J. Harnad in 2008 and 2009. The first lecture series *Introduction to τ-functions and their applications*, was intended to be a special preparatory series for the thematic year *Probabilistic Methods in Mathematical Physics* in 2008/09 at the Centre des recherches mathématiques. The second course, *Group Theory/Lie Groups* at Concordia University, was focusing on representation theory and group characters, from a unified Grassmannian viewpoint.

The goal of the present course is to give a minimal introduction to the notion of τ-functions via finite and infinite dimensional Grassmannians, and demonstrate through numerous examples how this τ-function appears naturally in many disguises in classical and quantum integrable systems, random matrices, random processes, and also in the theory of Riemann surfaces, group representations, symmetric functions, and combinatorics.

Time frame

April-May 2014 (6-7 weeks)
2×2 hours per week

Tentative lecture plan

Week 1. τ-functions on finite dimensional Grassmannians

- Quick review of the Grassmannian $Gr_k(\mathbb{C}^n)$
  Plücker embedding, Plücker relations. Dual Grassmannian. Labelling of rows by partitions and particle coordinates. Maya diagrams. Frobenius notation for partitions. Affine coordinates, generalized Giambelli formula.

- The τ-function, as a distinguished Plücker coordinate
  Abelian group action on $Gr_k(\mathbb{C}^n)$ generated by the powers of a shift matrix. Definition of the τ-function. Cauchy–Binet identity, Schur functions as Plücker coordinates, Schur function expansion of the τ-function.

- Symmetric functions
  Elementary, complete and power sum symmetric functions, Miwa variables. Basic identities for Schur functions, Jacobi-Trudi formulae, Giambelli identity. Cauchy–Littlewood identity. Schur functions evaluated at special values.

Week 2. Hirota bilinear equations and the KP hierarchy

- τ-function, as a generating function for Plücker coordinates
  Hirota bilinear operators. Plücker relations and Hirota bilinear equations for the τ-function. The KP equation.

- KP hierarchy
  Pseudo-differential operators. Baker–Akhiezer function, Sato formula. Wronskian and soliton solutions to the KP hierarchy.

Week 3. Some applications and the fermionic description of τ-functions

- Applications to group representation theory: Dimensions of irreducible representations of the symmetric group and the general linear group. Plancherel measure.
- Applications to symmetric functions
  Bases for the space of symmetric polynomials. Pieri formula. Skew Schur functions, Littlewood–Richardson coefficients.
- Fermionic formalism
  Finite dimensional fermionic Fock space, expectation values. Quantized flows, basic properties. Boson-fermion correspondence.

Week 4. Infinite dimensional Grassmannians and KP

- Fredholm determinants
  Finite rank perturbations of the identity. Weinstein–Aronszajn formula.
- Sato–Segal–Wilson Grassmannian in infinite dimensions
  Plücker coordinates as Fredholm determinants. The abelian group action associated to KP flows. The τ-function as a Fredholm determinant. Schur function expansion.
- Fermionic formalism in infinite dimensions
  Fermionic Fock space, vacuum expectation values, Wick theorem. Free fields.
- Related other integrable hierarchies
  Geometric constraints on the Grassmannian leading to the KdV and Gelfand–Dickey hierarchies

Week 5. Finite dimensional τ- functions, convolution symmetries

- Finite dimensional τ- functions
  Different finite dimensional group actions leading to KP dynamics. Gekhtman-Kasman formulae.
- Convolution symmetries
  The action of the group of non-singular diagonal matrices on the Grassmannian. Convolution interpretation. Convolution symmetries and Schur function expansions.

Week 6. Applications

- Hermitian matrix models and Toda lattice
  Hänkel determinants of moments of a measure, orthogonal polynomials. Matrix model partition functions. Christoffel–Darboux kernels. Riemann–Hilbert characterization of the orthogonal polynomials. Toda lattice hierarchy.
- Random partitions
  Toeplitz determinants, Gessel's theorem. Schur processes, z-measures. Limiting shapes and local asymptotics for large partitions.
- Calogero–Moser system
  Pole dynamics for certain rational solutions to KP. Wilson's adelic Grassmannian.
- Six-vertex model
  Korepin–Izergin determinant, associated matrix models, wheel conditions.

Time permitting

- Vertex algebra formalism
- Integrable kernels
- Block-Toeplitz determinants and loop groups
- Solutions to KP associated with algebraic curves, θ-function formulae (Krichever's solutions)
- BKP and DKP hierarchies

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