Gluing of Geometric PDEs: Obstructions vs. Constructions for Minimal Surfaces & Mean Curvature Flow Solitons

Abstract The goal of these two 90 minutes talks will be to (i) Give an introduction to mean curvature flow (the steepest descent flow for the area of hypersurfaces), and singularity formation in this flow, and to (ii) Introduce and explain in some detail the specific gluing techniques for nonlinear partial differential equations, developed in the context of minimal surfaces (e.g. mean curvature $H = 0$), which ultimately lead to construction of new examples of singularity models (solitons).

The first lecture will cover preliminaries on mean curvature flow and self-similar solutions to the flow: Maximum principles, monotonicity, parabolic blow-up, solitons and regularity. We will then describe some of the basic features of the partial differential equations that solitons satisfy, with focus mostly on self-shrinkers and their role in modeling the finite time singularities of the flow, and the relation to minimal surfaces. [For a very basic introduction to the topics, see f.ex. Colding & Minicozzi, arXiv:1102.1411. For a more solid/technical reference text for mean curvature flow, see f.ex. Ecker's book "Regularity theory for Mean Curvature Flow"]. As will be clear, for graphs $(p, u(p))$, where $p \in \mathbb{R}^n$, the nonlinear PDE for a self-shrinking soliton of the flow is

$$(\dagger) \quad \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \frac{1}{2}\frac{p\cdot\nabla u - u}{\sqrt{1+|\nabla u|^2}},$$

for which the local theory is completely classical. The global geometry and analysis of complete, embedded hypersurfaces $\Sigma^n \subseteq \mathbb{R}^n$ (both compact and non-compact with ends), which solve ($\dagger$), are thus the non-trivial aspects (and there are rigid obstructions!). I will include an overview of known examples and classification results for such soliton hypersurfaces.

In the second double lecture I will work through a gluing construction (with Kapouleas and Kleene) which yielded new complete, embedded, self-shrinkers Σ_g^2 of high genus g, in $\mathbb{R}^3$ (as conjectured from numerics by Tom Ilmanen, and others, in the early 90's), by fusing known low-genus examples. For this, I will first cover some basic geometric and analytical generalities of such desingularization constructions, then concentrate on some specific features of self-shrinkers. For example, the analysis in the situation with non-compact ends is complicated by the unbounded geometry. Therefore, certain Schrödinger operators with fast growth of the coefficients need to be understood well via geometric "Liouville" or "bubbling" results, which amounts to a certain decomposition into a compact model space plus decaying remainders. This in turn enables the construction of the resolvent of the stability operator, and controlling the higher order terms present in the nonlinear PDE in appropriate weighted Hölder spaces.